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On the Achievable Rate of Stationary Rayleigh Flat-Fading Channels with Gaussian Input Distribution

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Abstract

For a Gaussian input distribution, we investigate the achievable rate of a stationary Rayleigh flat-fading channel under the assumption of unknown channel state information at transmitter and receiver side. The law of the channel is presumed to be known to the receiver. In addition, we assume the power spectral density of the fading process to be compactly supported. The contribution of the present paper is the derivation of an upper bound on the achievable rate for the special case of a rectangular power spectral density depending on the SNR and the spread of the power spectral density. For comparison, we also give a lower bound on the achievable rate which is already known from [1] and holds for an arbitrary power spectral density.

1. INTRODUCTION

The capacity of fading channels where the channel state information is unknown to the receiver has received a lot of attention in the recent literature, e.g., [2, 3, 4, 5, 6] as this scenario applies to many realistic mobile communication systems.

In this paper, we consider a discrete-time single-input single-output stationary Rayleigh flat-fading channel with temporal correlation. We assume that the channel state information is unknown to the transmitter and the receiver, while the receiver is aware of the channel law. We investigate its achievable rate while restricting to Gaussian input distributions. We consider a stationary zero-mean jointly proper Gaussian [7] fading process. In addition, we assume that the power spectral density (PSD) of the fading process has compact support.

The contribution of the present paper is the derivation of an upper bound on the achievable rate for the special case of a rectangular PSD, depending on the SNR and the spread of the PSD. Especially, the therefor used lower bound on the conditional output entropy rate $h'(\mathbf{y}|\mathbf{x})$ for a rectangular PSD is to the best of our knowledge new. The assumption of a rectangular PSD is usually made in typical communication system design. For comparison, we give a lower bound on the achievable rate, holding for an arbitrary PSD with compact support, which is already known from [1].

In contrast to existing bounds on the capacity for flat-fading channels which are asymptotic, e.g., [2] for the high SNR regime, or tight in a specific SNR regime, e.g., [5] for the low SNR regime and a peak power constraint, our aim is to get bounds on the achievable rate that are valid over a wide range of the SNR. As the evaluation of the capacity, including the maximization of the mutual information over the input distribution, is very difficult, we restrict to Gaussian input distributions, which are capacity-achieving in case of perfect channel state information. However, they are not optimal for unknown channel state information at the receiver [8]. In [9], bounds on the mutual information with Gaussian input distributions have been derived for a Gauss-Markov fading channel, whose PSD has an unbounded support. The results in [9] indicate that at moderate SNR and/or slow fading, Gaussian inputs still work well.

2. SYSTEM MODEL

We consider an ergodic discrete-time jointly proper Gaussian flat-fading channel, whose I/O relation is given by

$$\mathbf{y} = \mathbf{H} \cdot \mathbf{x} + \mathbf{n} = \mathbf{X} \cdot \mathbf{h} + \mathbf{n} \tag{1}$$

where the the channel output vector is defined by $\mathbf{y} = [y_1, \dots, y_N]^T$ and y_k is the channel output at time instant k. Analogously, the channel input vector is given by $\mathbf{x} = [x_1, \dots, x_N]^T$, and the noise vector is

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$\mathbf{n} = [n_1, \ldots, n_N]^T$. The matrix $\mathbf{H}$ is diagonal and defined as $\mathbf{H} = \text{diag}(\mathbf{h})$ with the vector $\mathbf{h} = [h_1, \ldots, h_N]^T$ describing the channel fading process. Here the diag$(\cdot)$ operator generates a diagonal matrix whose diagonal elements are given by the argument vector. The diagonal matrix $\mathbf{X}$ is given by $\mathbf{X} = \text{diag}(\mathbf{x})$. The quantity N is the number of considered symbols. Later on, we investigate the case $N \to \infty$ to evaluate the achievable rate.

We assume that the noise n_k is a sequence of i.i.d. circularly symmetric complex Gaussian random variables of zero mean and variance σ_n^2, i.e., $\mathbf{R}_n = \text{E}[\mathbf{n}\mathbf{n}^H] = \sigma_n^2 \cdot \mathbf{I}_N$, where $\mathbf{I}_N$ is the $N \times N$ identity matrix. The channel fading process $\{h_k\}$ is zero-mean jointly proper Gaussian. It is time selective and is characterized by its autocorrelation function $r_h(l) = \text{E}[h_{k+l} \cdot h_k^*]$. Its variance is given by $r_h(0) = \sigma_h^2$.
The PSD of the channel fading process is defined as

$$S_h(f) = \sum_{m=-\infty}^{\infty} r_h(m) e^{-j2\pi m f}, \quad |f| \leq 0.5 \qquad (2)$$

where we assume that the PSD exists. For a jointly proper Gaussian process this implies ergodicity [10]. Because of the limitation of the velocity of the transmitter, the receiver, and of objects in the environment, the spread of the PSD is limited, and we assume it to be compactly supported within the interval $[-f_d, f_d]$, with $0 < f_d \leq 0.5$, i.e., $S_h(f) = 0$ for $f \notin [-f_d, f_d]$. The parameter f_d corresponds to the maximum Doppler shift and thus indicates the dynamics of the channel. To ensure ergodicity, we exclude the case $f_d = 0$.

The temporal correlation of the fading process can be expressed by the correlation matrix $\mathbf{R}_h = \text{E}[\mathbf{h}\mathbf{h}^H]$, which has a Hermitian Toeplitz structure. The channel input $\mathbf{x}$ is jointly proper Gaussian and its elements are independent, yielding $\mathbf{R}_x = \text{E}[\mathbf{x}\mathbf{x}^H] = \sigma_x^2 \cdot \mathbf{I}_N$. The processes $\{h_k\}$, $\{x_k\}$, and $\{n_k\}$ are assumed to be mutually independent.

The mean SNR is given by

$$\rho = \frac{\sigma_x^2 \sigma_h^2}{\sigma_n^2}. \qquad (3)$$

3. BOUNDS ON THE ACHIEVABLE RATE

As the PSD of the fading process exists (2) and as the channel fading process is jointly proper Gaussian, the channel fading process is ergodic. Therefore, operational and information theoretic capacity coincide [10]. This allows us to base our following derivations on the concept of the ergodic capacity.

In this work, we restrict to proper Gaussian input distributions and do not maximize the mutual information $\mathcal{I}(\mathbf{y}; \mathbf{x})$ over the input distribution. Therefore, we do not use the term capacity, but the term *achievable rate*, given by

$$R = \lim_{N \to \infty} \frac{1}{N} \mathcal{I}(\mathbf{y}; \mathbf{x}) = \mathcal{I}'(\mathbf{y}; \mathbf{x}) \qquad (4)$$

being equivalent to the rate of the mutual information.

The mutual information rate can be expanded as

$$\mathcal{I}'(\mathbf{y}; \mathbf{x}) = \mathcal{I}'(\mathbf{y}; \mathbf{x}|\mathbf{h}) - \mathcal{I}'(\mathbf{x}; \mathbf{h}|\mathbf{y}) \qquad (5)$$

$$= h'(\mathbf{y}) - h'(\mathbf{y}|\mathbf{x}) \qquad (6)$$

where $\mathcal{I}'(\mathbf{y}; \mathbf{x}|\mathbf{h})$ in (5) is the mutual information rate in case the channel is known at the receiver, i.e., the mutual information rate of the coherent channel, and $\mathcal{I}'(\mathbf{x}; \mathbf{h}|\mathbf{y})$ is the penalty due to the channel uncertainty. We will make use of the separation in (6) to derive an upper and a lower bound on $\mathcal{I}'(\mathbf{y}; \mathbf{x})$. In (6) $h'(\cdot)$ indicates the differential entropy rate, i.e., $h'(\cdot) = \lim_{N\to\infty} \frac{1}{N} h(\cdot)$, where $h(\cdot)$ is the differential entropy.

3.1. The Received Signal Entropy Rate $h'(\mathbf{y})$

3.1.1. Lower bound on $h'(\mathbf{y})$

The mutual information with perfect channel state information at the receiver can be upper-bounded by

$$\mathcal{I}(\mathbf{y}; \mathbf{x}|\mathbf{h}) = h(\mathbf{y}|\mathbf{h}) - h(\mathbf{y}|\mathbf{h}, \mathbf{x}) \leq h(\mathbf{y}) - h(\mathbf{y}|\mathbf{h}, \mathbf{x}) \qquad (7)$$

where we use the fact that conditioning reduces entropy. Thus, we can lower-bound $h'(\mathbf{y})$ by

$$h'(\mathbf{y}) \geq \mathcal{I}'(\mathbf{y}; \mathbf{x}|\mathbf{h}) + h'(\mathbf{y}|\mathbf{h}, \mathbf{x}). \qquad (8)$$

The mutual information rate in case the channel is known at the receiver, i.e., the first term on the RHS of (8), is given by

$$\mathcal{I}'(\mathbf{y}; \mathbf{x}|\mathbf{h}) = \frac{1}{N} \text{E}_{\mathbf{h}} \left[\text{E}_{\mathbf{y},\mathbf{x}} \left\{ \log \left(\frac{p(\mathbf{y}|\mathbf{h}, \mathbf{x})}{p(\mathbf{y}|\mathbf{h})} \right) \middle| \mathbf{h} \right\} \right]$$

$$= \int_0^\infty \log(\rho \cdot z + 1) e^{-z} dz \qquad (9)$$

and is independent of the temporal correlation of the channel, see, e.g., [3].

The second term on the RHS of (8) originates from AWGN and, thus, can be calculated as

$$h'(\mathbf{y}|\mathbf{h}, \mathbf{x}) = \log(\pi e \sigma_n^2). \qquad (10)$$

Using (8), (9), and (10) we get a lower bound $h'_L(\mathbf{y})$ on the entropy rate

$$h'(\mathbf{y}) \geq h'_L(\mathbf{y})$$

$$= \int_0^\infty \log(\rho \cdot z + 1) e^{-z} dz + \log(\pi e \sigma_n^2). \qquad (11)$$

3.1.2. Upper bound on $h'(\mathbf{y})$

In this section we give an upper bound on the entropy rate $h'(\mathbf{y})$. First, we make use of the fact that the entropy $h(\mathbf{y})$ of a complex random vector $\mathbf{y}$ of dimension N with nonsingular correlation matrix $\mathbf{R}_y = \mathrm{E}[\mathbf{y}\mathbf{y}^H]$ is upper-bounded by [7]

$$h(\mathbf{y}) \leq \log\left[(\pi e)^N \det(\mathbf{R}_y)\right]. \quad (12)$$

Due to the independency of the transmit symbols, the correlation matrix becomes

$$\mathbf{R}_y = (\sigma_x^2\sigma_h^2 + \sigma_n^2)\mathbf{I}_N. \quad (13)$$

Hence, an upper bound $h'_U(\mathbf{y})$ on the entropy rate $h'(\mathbf{y})$ is given by

$$h'(\mathbf{y}) \leq \log\left(\pi e\left(\sigma_x^2\sigma_h^2 + \sigma_n^2\right)\right) = h'_U(\mathbf{y}). \quad (14)$$

3.1.3. Tightness of the upper bound and the lower bound on $h'(\mathbf{y})$

The difference between upper bound $h'_U(\mathbf{y})$ and lower bound $h'_L(\mathbf{y})$ is given by

$$\begin{aligned}\Delta_{h'(\mathbf{y})} &= h'_U(\mathbf{y}) - h'_L(\mathbf{y}) \\ &= \log(\rho+1) - \int_0^\infty \log(\rho\cdot z+1)\, e^{-z} dz \quad (15)\end{aligned}$$

where ρ is the mean SNR defined in (3).

For $\rho \to 0$ the difference of the upper bound and the lower bound converges to zero, $\lim_{\rho\to 0}\Delta_{h'(\mathbf{y})} = 0$. For $\rho \to \infty$ the difference is given by

$$\lim_{\rho\to\infty}\Delta_{h'(\mathbf{y})} = \gamma \approx 0.57721 \quad [\text{nat}]. \quad (16)$$

where γ is the Euler constant. The limit in (16) can be found in [8].

It can be shown that $\Delta_{h'(\mathbf{y})}$ monotonically increases with the SNR. Thus, the difference $\Delta_{h'(\mathbf{y})}$ is bounded

$$0 \leq \Delta_{h'(\mathbf{y})} \leq \gamma \quad [\text{nat}]. \quad (17)$$

3.2. The Entropy Rate $h'(\mathbf{y}|\mathbf{x})$

3.2.1. Upper bound on $h'(\mathbf{y}|\mathbf{x})$

The probability density function of $\mathbf{y}$ conditioned on $\mathbf{x}$ is zero-mean proper Gaussian, thus its entropy is given by

$$h(\mathbf{y}|\mathbf{x}) = \mathrm{E}_{\mathbf{x}}\left[\log\left((\pi e)^N \det(\mathbf{R}_{y|x})\right)\right] \quad (18)$$

where the covariance matrix $\mathbf{R}_{y|x}$ is given by

$$\begin{aligned}\mathbf{R}_{y|x} &= \mathrm{E}_{\mathbf{h},\mathbf{n}}[\mathbf{y}\mathbf{y}^H] = \mathrm{E}_{\mathbf{h}}[\mathbf{X}\mathbf{h}\mathbf{h}^H\mathbf{X}^H] + \sigma_n^2\mathbf{I}_N \\ &= \mathbf{X}\mathbf{R}_h\mathbf{X}^H + \sigma_n^2\mathbf{I}_N. \quad (19)\end{aligned}$$

The channel correlation matrix can be decomposed by a spectral decomposition as

$$\mathbf{R}_h = \mathbf{U}\mathbf{\Lambda}_h\mathbf{U}^H \quad (20)$$

where the diagonal matrix $\mathbf{\Lambda}_h = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ contains the eigenvalues λ_i of $\mathbf{R}_h$ and the matrix $\mathbf{U}$ is unitary.

In the following we will upper-bound $h'(\mathbf{y}|\mathbf{x})$ in the same way as it is already known from [1]. The entropy $h(\mathbf{y}|\mathbf{x})$ is upper-bounded by

$$h(\mathbf{y}|\mathbf{x}) = \mathrm{E}_{\mathbf{x}}\left[\log\left((\pi e)^N \det\left(\mathbf{X}\mathbf{U}\mathbf{\Lambda}_h\mathbf{U}^H\mathbf{X}^H + \sigma_n^2\mathbf{I}_N\right)\right)\right] \quad (21)$$

$$\begin{aligned}&\overset{(a)}{=} \mathrm{E}_{\mathbf{x}}\log\left[\det\left(\frac{\mathbf{X}^H\mathbf{X}\mathbf{U}\mathbf{\Lambda}_h\mathbf{U}^H}{\sigma_n^2} + \mathbf{I}_N\right)\right] + N\log(\pi e\sigma_n^2) \\ &\overset{(b)}{\leq} \log\mathrm{E}_{\mathbf{x}}\left[\det\left(\frac{\mathbf{X}^H\mathbf{X}\mathbf{U}\mathbf{\Lambda}_h\mathbf{U}^H}{\sigma_n^2} + \mathbf{I}_N\right)\right] + N\log(\pi e\sigma_n^2) \\ &\overset{(c)}{=} \log\det\left(\frac{\sigma_x^2}{\sigma_n^2}\mathbf{U}\mathbf{\Lambda}_h\mathbf{U}^H + \mathbf{I}_N\right) + N\log(\pi e\sigma_n^2) \\ &= \sum_{i=1}^N \log\left(\frac{\sigma_x^2}{\sigma_n^2}\lambda_i + 1\right) + N\log(\pi e\sigma_n^2) \quad (22)\end{aligned}$$

where for (a) the following relation is used

$$\det(\mathbf{AB} + \mathbf{I}) = \det(\mathbf{BA} + \mathbf{I}) \quad (23)$$

as $\mathbf{AB}$ has the same eigenvalues as $\mathbf{BA}$ for $\mathbf{A}$ and $\mathbf{B}$ being square matrices, [11, Th. 1.3.20]. (b) is due to Jensen's inequality and the concavity of the log-function. The equality (c) is due to the statistical independence of the transmit symbols and can be shown by using the Laplacian expansion by minors[1].

To calculate the bound for the entropy rate $h'(\mathbf{y}|\mathbf{x})$ we consider the case $N \to \infty$, i.e., the dimension of the matrix $\mathbf{\Lambda}_h$ grows without bound. As $\mathbf{R}_h$ is Hermitian Toeplitz, we can evaluate (22) using Szegö's theorem about the asymptotic eigenvalue distribution of Hermitian Toeplitz matrices [12]. Consequently

$$\lim_{N\to\infty}\frac{1}{N}\sum_{i=1}^N \log\left(\frac{\sigma_x^2}{\sigma_n^2}\lambda_i + 1\right) = \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(\frac{\sigma_x^2 S_h(f)}{\sigma_n^2} + 1\right) df. \quad (24)$$

Hence, we get the following upper bound

$$\begin{aligned}h'(\mathbf{y}|\mathbf{x}) &\leq h'_U(\mathbf{y}|\mathbf{x}) \\ &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(\frac{\sigma_x^2 S_h(f)}{\sigma_n^2} + 1\right) df + \log(\pi e\sigma_n^2). \quad (25)\end{aligned}$$

[1](b) and (c) can also be derived jointly using that $\log\det(\cdot)$ is concave on the set of positive definite matrices.

3.2.2. Lower bound on $h'(\mathbf{y}|\mathbf{x})$ for a rectangular PSD

In this section we give a lower bound for the entropy rate $h'(\mathbf{y}|\mathbf{x})$ for the special case of a rectangular PSD, which is used as an approximation to the actual PSD. In addition, in typical system design the channel dynamic is characterized by the sole parameter f_d and the assumption of a rectangular PSD.

We now assume that the eigenvalues λ_i of $\mathbf{R}_h$ are given by

$$\lambda_i = \begin{cases} \frac{\sigma_h^2}{2\cdot f_d} & 1 \leq i \leq 2f_d \cdot N \\ 0 & \text{otherwise} \end{cases} . \quad (26)$$

For the case of $N \to \infty$ this corresponds to a rectangular PSD.

With (26) the entropy given in (21) can be transformed to

$$h(\mathbf{y}|\mathbf{x}) \overset{(a)}{=} \mathrm{E}_{\mathbf{x}}\left[\log\left((\pi e)^N \det\left(\mathbf{\Lambda}_h \mathbf{U}^H \mathbf{X}^H \mathbf{X} \mathbf{U} + \sigma_n^2 \mathbf{I}_N\right)\right)\right]$$
$$\overset{(b)}{=} \mathrm{E}_{\mathbf{x}}\left[\log\det\left(\frac{\sigma_h^2 \tilde{\mathbf{U}}^H \mathbf{X}^H \mathbf{X} \tilde{\mathbf{U}}}{2f_d\sigma_n^2} + \mathbf{I}_{\lfloor 2f_d N\rfloor}\right)\right] + N\log(\pi e \sigma_n^2) \quad (27)$$

where for (a) we used (23). Furthermore, for (b) we used (26) and the matrix $\tilde{\mathbf{U}}$ is given by

$$\tilde{\mathbf{U}} = \left[\mathbf{u}_1, \ldots, \mathbf{u}_{\lfloor 2f_d N\rfloor}\right] \in \mathbb{C}^{N \times \lfloor 2f_d N\rfloor} \quad (28)$$

where $\mathbf{u}_i$ are the orthonormal columns of the unitary matrix $\mathbf{U}$. Now we apply the following inequality given in [13].

Lemma 1: Let $\mathbf{A} \in \mathbb{C}^{m\times n}$ with orthonormal rows and $m \leq n$. Then

$$\log\det\left(\mathbf{A}\mathrm{diag}\left(p_1, \ldots, p_n\right)\mathbf{A}^H\right)$$
$$\geq \mathrm{tr}\left[\mathbf{A}\mathrm{diag}(\log p_1, \ldots, \log p_n)\mathbf{A}^H\right] \quad (29)$$

if $p_1, \ldots, p_n > 0$. Here tr denotes the trace of a matrix. With Lemma 1 we can lower-bound (27) so that

$$h(\mathbf{y}|\mathbf{x}) \geq N\log(\pi e\sigma_n^2)$$
$$+\mathrm{E}_{\mathbf{x}}\left[\mathrm{tr}\left[\tilde{\mathbf{U}}^H \mathrm{diag}\left(\log\left(\frac{\sigma_h^2|x_1|^2}{2f_d\sigma_n^2}+1\right), .., \log\left(\frac{\sigma_h^2|x_N|^2}{2f_d\sigma_n^2}+1\right)\right)\tilde{\mathbf{U}}\right]\right]$$
$$= N\log(\pi e\sigma_n^2)$$
$$+\mathrm{tr}\left[\tilde{\mathbf{U}}^H \mathrm{diag}\left(\mathrm{E}_{\mathbf{x}}\log\left(\frac{\sigma_h^2|x_1|^2}{2f_d\sigma_n^2}+1\right), .., \mathrm{E}_{\mathbf{x}}\log\left(\frac{\sigma_h^2|x_N|^2}{2f_d\sigma_n^2}+1\right)\right)\tilde{\mathbf{U}}\right]$$
$$\overset{(a)}{=} \sum_{k=1}^{\lfloor 2f_d N\rfloor} \mathrm{E}_x \log\left(\frac{\sigma_h^2}{2f_d\sigma_n^2}|x|^2+1\right) + N\log(\pi e\sigma_n^2) \quad (30)$$

where (a) results because all x_k are identically distributed and because the columns of $\tilde{\mathbf{U}}$ are orthonormal.

For $N \to \infty$ we get

$$h'(\mathbf{y}|\mathbf{x}) = \lim_{N\to\infty} \frac{1}{N} h(\mathbf{y}|\mathbf{x})$$
$$\geq \lim_{N\to\infty} \frac{1}{N} \sum_{k=1}^{\lfloor 2f_d N\rfloor} \mathrm{E}_x \log\left(\frac{\sigma_h^2}{2f_d\sigma_n^2}|x|^2+1\right) + \log(\pi e\sigma_n^2)$$
$$= 2f_d \mathrm{E}_x \log\left(\frac{\sigma_h^2}{2f_d\sigma_n^2}|x|^2+1\right) + \log(\pi e\sigma_n^2)$$
$$= 2f_d \int_0^\infty \log\left(\frac{\rho}{2f_d}z+1\right)e^{-z}dz + \log(\pi e\sigma_n^2) = h'_L(\mathbf{y}|\mathbf{x}). \quad (31)$$

3.2.3. Tightness of upper and lower bound on $h'(\mathbf{y}|\mathbf{x})$

The difference between the upper bound and the lower bound on $h'(\mathbf{y}|\mathbf{x})$ in case of a rectangular PSD is given by

$$\Delta_{h'(\mathbf{y}|\mathbf{x})} = h'_U(\mathbf{y}|\mathbf{x}) - h'_L(\mathbf{y}|\mathbf{x})$$
$$= 2f_d \int_0^\infty \log\left(\frac{\frac{\rho}{2f_d}+1}{\frac{\rho}{2f_d}z+1}\right)e^{-z}dz. \quad (32)$$

For asymptotically small Doppler frequencies we get $\lim_{f_d\to 0}\Delta_{h'(\mathbf{y}|\mathbf{x})} = 0$ independent of the SNR ρ. Concerning the dependency of $\Delta_{h'(\mathbf{y}|\mathbf{x})}$ on the SNR, it can be shown that $\lim_{\rho\to 0}\Delta_{h'(\mathbf{y}|\mathbf{x})} = 0$ independent of f_d. For asymptotically high SNR, the difference is bounded by $\lim_{\rho\to\infty}\Delta_{h'(\mathbf{y}|\mathbf{x})} = 2f_d\gamma \approx 2f_d \cdot 0.57721$. As $\Delta_{h'(\mathbf{y}|\mathbf{x})}$ is monotonic in the SNR, cf. Section 3.1.3, it is bounded by

$$0 \leq \Delta_{h'(\mathbf{y}|\mathbf{x})} \leq \gamma 2f_d \quad [\text{nat}] \quad (33)$$

enabling a characterization of the tightness of the upper bound and the lower bound on $h'(\mathbf{y}|\mathbf{x})$ for different f_d.

3.3. The Achievable Rate

3.3.1. Upper and lower bound

Based on the upper and lower bounds on $h'(\mathbf{y})$ and $h'(\mathbf{y}|\mathbf{x})$, we are now able to give an upper bound and a lower bound on the achievable rate (4)

$$\mathcal{I}'_{U/L}(\mathbf{y};\mathbf{x}) = h'_{U/L}(\mathbf{y}) - h'_{L/U}(\mathbf{y}|\mathbf{x}). \quad (34)$$

The lower bound holds for an arbitrary PSD of the channel fading process, whereas the upper bound is only valid for a rectangular PSD, as we use this restriction for the derivation of $h'_L(\mathbf{y}|\mathbf{x})$.

For a rectangular PSD of the channel fading process the upper and the lower bound on the achievable rate

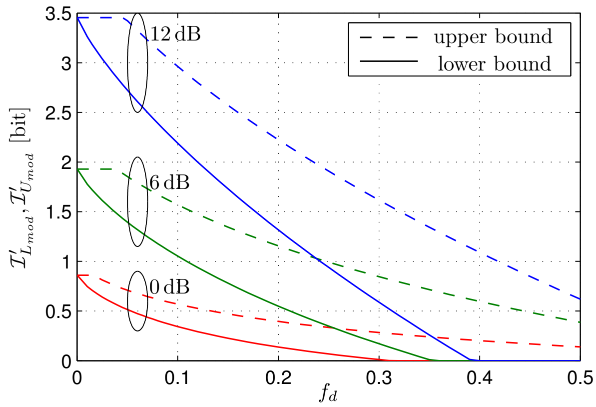

Figure 1: Upper and lower bound for the achievable rate under the assumption of a Gaussian input distribution of a Rayleigh flat-fading channel with a rectangular PSD depending on f_d

are given by

$$\mathcal{I}'_L(\mathbf{y};\mathbf{x}) = \int_0^\infty \log(\rho z + 1)\, e^{-z} dz - 2f_d \log\left(\frac{\rho}{2f_d} + 1\right) \tag{35}$$

$$\mathcal{I}'_U(\mathbf{y};\mathbf{x}) = \log(\rho + 1) - 2f_d \int_0^\infty \log\left(\frac{\rho}{2f_d} z + 1\right) e^{-z} dz. \tag{36}$$

Furthermore, we know that the mutual information rate in case of perfect channel state information at the receiver (9) always upper-bounds the mutual information rate in the absence of channel state information, $\mathcal{I}'(\mathbf{y};\mathbf{x}) \leq \mathcal{I}'(\mathbf{y};\mathbf{x}|\mathbf{h})$. Therefore, we can modify the upper bound as follows

$$\mathcal{I}'_{U_{mod}}(\mathbf{y};\mathbf{x}) = \min\{\mathcal{I}'_U(\mathbf{y};\mathbf{x}), \mathcal{I}'(\mathbf{y};\mathbf{x}|\mathbf{h})\}. \tag{37}$$

On the other hand, the mutual information rate must be non-negative. Thus, we modify the lower bound to

$$\mathcal{I}'_{L_{mod}}(\mathbf{y};\mathbf{x}) = \max\{\mathcal{I}'_L(\mathbf{y};\mathbf{x}), 0\}. \tag{38}$$

Fig. 1 shows the upper bound (37) and the lower bound (38) on the achievable rate for a rectangular PSD of the channel fading process as a function of the channel dynamic, which is characterized by f_d.

3.3.2. Tightness of upper and lower bound on $\mathcal{I}'(\mathbf{y};\mathbf{x})$

For $f_d \to 0$ the lower bound $\mathcal{I}'_L(\mathbf{y};\mathbf{x})$ is equivalent to the mutual information rate in case of perfect channel knowledge (9)

$$\lim_{f_d \to 0} \mathcal{I}'_L(\mathbf{y};\mathbf{x}) = \mathcal{I}'(\mathbf{y};\mathbf{x}|\mathbf{h}). \tag{39}$$

This corresponds to the physical interpretation that a channel that changes arbitrarily slowly can be estimated arbitrarily well, and therefore, the penalty term $\mathcal{I}'(\mathbf{x};\mathbf{h}|\mathbf{y})$ in (5) approaches zero.

The difference between the upper bound $\mathcal{I}'_U(\mathbf{y};\mathbf{x})$ and the lower bound $\mathcal{I}'_L(\mathbf{y};\mathbf{x})$ is given by

$$\Delta_{\mathcal{I}'(\mathbf{y};\mathbf{x})} = \mathcal{I}'_U(\mathbf{y};\mathbf{x}) - \mathcal{I}'_L(\mathbf{y};\mathbf{x}) = \Delta_{h'(\mathbf{y})} + \Delta_{h'(\mathbf{y}|\mathbf{x})}.$$

As

$$\lim_{\rho \to 0} \Delta_{\mathcal{I}'(\mathbf{y};\mathbf{x})} = 0$$
$$\lim_{\rho \to \infty} \Delta_{\mathcal{I}'(\mathbf{y};\mathbf{x})} = \gamma(1 + 2f_d) \tag{40}$$

and as $\Delta_{\mathcal{I}'(\mathbf{y};\mathbf{x})}$ monotonically increases with the SNR, the difference is bounded by

$$0 \leq \Delta_{\mathcal{I}'(\mathbf{y};\mathbf{x})} \leq \gamma(1 + 2f_d) \quad \text{[nat]}. \tag{41}$$

4. THE ASYMPTOTIC HIGH SNR LIMIT

In this section, we examine the slope of the achievable rate with a Gaussian input distribution over the SNR for asymptotically large SNRs depending on the channel dynamics. It can be shown that for a compactly supported PSD as defined in Section 2 the following relation holds

$$\lim_{\rho\to\infty} \frac{\partial \mathcal{I}'_L(\mathbf{y};\mathbf{x})}{\partial \log(\rho)} = \lim_{\rho\to\infty} \frac{\partial}{\partial \log(\rho)} \Bigg[\int_{z=0}^{\infty} \log(\rho z+1) e^{-z} dz - \int_{-\frac{1}{2}}^{\frac{1}{2}} \log\left(\frac{S_h(f)}{\sigma_h^2}\rho + 1\right) df \Bigg]$$
$$= 1 - 2f_d \tag{42}$$

as $S_h(f) \neq 0$ for $|f| \leq f_d$.

The upper bound $\mathcal{I}'_U(\mathbf{y};\mathbf{x})$ holds only for the special case of a rectangular PSD of the channel fading process. For this case the difference between the upper bound $\mathcal{I}'_U(\mathbf{y};\mathbf{x})$ and the lower bound $\mathcal{I}'_L(\mathbf{y};\mathbf{x})$ converges to a constant for high SNR, cf. (40). Thus, both bounds must have the same asymptotic high SNR slope and we conjecture that the achievable rate $\mathcal{I}'(\mathbf{y};\mathbf{x})$ is also characterized by the same asymptotic SNR slope.

5. COMPARISON TO ASYMPTOTES IN [2]

Fig. 2 shows the comparison of the bounds (37) and (38) for the achievable rate in case of a Gaussian input distribution towards the high SNR asymptotes for the upper bound and the lower bound on the capacity in the corresponding pre-log case given in [2]. Here it has to be stated that the bounds on capacity given in [2] are based on a peak power constraint in addition to the average power constraint that holds also for the

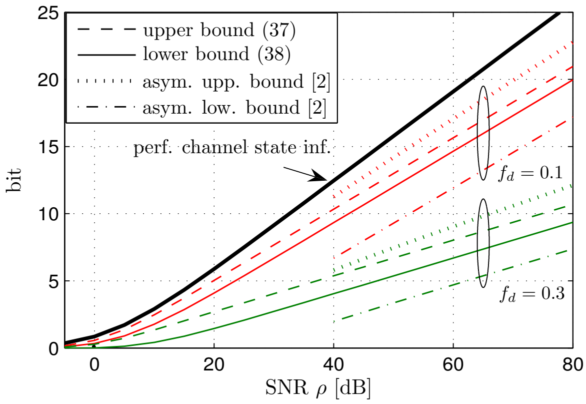

Figure 2: Comparison of bounds on the achievable rate (37) and (38) with asymptotic capacity bounds in [2, eq. (33) and (47)] (The asymptotic upper bound only holds for $\rho \to \infty$ as we neglect the term $o(1)$ in [2, eq.(33)], which approaches zero for $\rho \to \infty$.)

bounds on the achievable rate in case of a Gaussian input distribution in the present work. Therefore, this comparison is only of qualitative nature. Beside the power constraints [2] does not constrain the input distribution and therefore delivers bounds for the capacity. For asymptotically large SNR the bounds on the achievable rate with Gaussian inputs and the bounds in [2] have the same slope, which corresponds to the pre-log behavior of capacity as described in [2].

6. SUMMARY

In this paper, we have derived a new non-asymptotic upper bound on the achievable rate with Gaussian input distribution for a stationary Rayleigh flat-fading channel and a rectangular PSD of the channel fading process. The channel state information is assumed to be unknown to the transmitter and the receiver, while the receiver is aware of the channel law. For comparison, we have also given a lower bound on the achievable rate for Gaussian input distributions, which is already known from [1].

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