Nonlinear dynamics

- Celestial mechanics
  
  Newton

  \[ \dot{r} = -\frac{GM}{r^2}, \quad \dot{\phi} = \frac{\text{const}}{r} \]

  \[ \dot{x} = f(x) \]

- Electric circuits
- Control systems in engineering
- Biological systems
- Economical models.

⇒ Use a geometric way of thinking.
Basic concepts

- Phase space \( \mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \)

- Trajectory \( \mathbf{x}(t) \)

- Flow field \( \dot{\mathbf{x}} = f(\mathbf{x}) \)
  
  (no explicit time dep: autonomous)

- Fixed point

  \[ \mathbf{x}(t) = \mathbf{x}_0 \implies f(\mathbf{x}_0) = 0. \]

- Stability

  \[ \mathbf{x}_0 = \text{stable fixed point} \]

  \[ \forall \text{ trajectories starting close to } \mathbf{x}_0 \text{ converge to } \mathbf{x}_0. \]

[We will use different definitions for stability later]
Example:

Overdamped motion in a washboard potential

\[ \dot{x} = -\frac{\partial U}{\partial x}, \quad U = U_0 \cos kx. \]

Stable fixed point

Unstable

Bassin-of-attraction
Example logistic growth

\[ \dot{N} = r N \left( 1 - \frac{N}{K} \right) \]
Existence and uniqueness.

\[ \Rightarrow \text{ trajectories in phase space cannot cross.} \]

Blow-up of solutions

\[ \frac{dx}{dt} = x^2 = 1 + x^2, \quad x(0) = 0. \]

\[ \int_0^{x(t)} \frac{dx}{1+x^2} = \int_0^t \frac{dt}{1} \]

\[ \tan^{-1} x = \frac{t}{1} (\pi + c) \]

\[ x = \tan t / \pi \]

\[ t = \pi \]
Existence and uniqueness
(Picard-Lindelof Theorem
Caucy-Lipschitz Theorem)

$$\dot{x} = f(x), \quad x(t_0) = x_0.$$

$f = \text{Lipschitz continuous e.g. } \|f'/1\| \leq M \Rightarrow$

There's unique solution for $[t_0, t_0 + \varepsilon].$

Idea of proof:
Construct iterative approximations

Way #1:

$$x(n + 1) = x_n$$
End Scheme:

$$x_n = x_{n-1} + f(x_{n-1}) \Delta t$$
$$\Rightarrow \text{let } \Delta t \to 0.$$

Way #2:

$$L: \{ \xi(t) \} \to \{ \xi_0 + \int_{t_0}^{t} f(x(t)) dt \}$$
Show that $\|L\| < 1$ on a suitable function space (and smaller)

$\Rightarrow$ Banach fixed point theorem guarantees unique fixed point.
Example: Non-uniqueness of solutions.

The empty bucket

- Mass conservation
  \[ \dot{h} = -av \]
- Energy conservation
  \[ \frac{d}{dt} \frac{1}{2} v^2 = g \cdot h \cdot \text{dm} \]

We know:

\[ h(t = 0) = 0 \]

When was

\[ h(t_0) = h_0 \text{ for } t < 0? \]

\[ \dot{v} = \sqrt{2gh} \]

\[ \dot{h} = \frac{\dot{v}}{A} = -\frac{a\sqrt{2gh}}{A} \]

\[ \int_{h_0}^h \frac{dh}{\sqrt{h}} = -\frac{a}{A} \sqrt{2g} \int_{t_0}^t \dot{v} dt \]

\[ -\frac{1}{2} \sqrt{2gh} \bigg|_{h_0}^0 = -\frac{a}{A} \sqrt{2g} \left( t - t_0 \right) \]

\[ h = \frac{(\dot{v})^2}{2g} \div \frac{1}{2} \left( t - t_0 \right)^2 \]

\[ \sqrt{\frac{2h_0}{g}} \cdot \frac{A}{a} = t - t_0 \]

\[ h_0 \rightarrow h \]
Celestial mechanics

2 bodies:

\[ m_1 \ddot{r}_1 = -\frac{\mu m_2}{r^2} \]
\[ m_2 \ddot{r}_2 = -\frac{\mu m_1}{r^2} \]
\[ r = r_1 - r_2 \]
\[ \mu = \frac{m_1}{m_1 + m_2} \]
\[ \mu \ddot{r} = -\frac{\mu}{m_1 + m_2} \]

\[ r = \frac{m_1}{m_1 + m_2} \]

1 body problem

3 body problem: would solve logistic problem [Poincaré, Nester]

Ferentia of motion:

- center of mass \( R \)
- energy \( E = E_{\text{kin}} + E_{\text{pot}} \)
- angular momentum \( L = m_1 r^2 \dot{\theta} \)

\( \Rightarrow \) motion in plane parallel to \( L \)
How many degrees of freedom remain?

\[ n=2 \quad 6n = 12 \]

Noether theorem:
- time invariance:
- rotational invariance:

\[ \frac{\dot{\theta}}{\mu r^2} = \text{(Kepler's 2nd law)} \]

\[ E = E_{\text{kin}} + E_{\text{pot}} \]
\[ = \frac{1}{2} \mu i^2 + \frac{1}{2} \mu r^2 \dot{\theta}^2 + \frac{8 m_1 m_2}{r} \]

\[ \Rightarrow \text{solve for } i \]
\[ i = \sqrt{\frac{2}{\mu}} \left( E - \frac{1}{2} \frac{L^2}{\mu r^2} + \frac{8 m_1 m_2}{r} \right)^{1/2} \]

\[ \Rightarrow \frac{\dot{\theta}}{i} = \frac{d\theta}{dr} \quad \text{is separable} \]

\[ \Rightarrow \text{Kepler's orbit} \]
\[ r = \frac{L}{\mu g m_1 m_2} (1 - e \cos \theta) \]
\[ l_0 = \frac{L}{\mu g m_1 m_2} \]
\[ e = \sqrt{1 + \frac{2 EL}{\mu g^2}} \]
\[ = \text{eccentricity} \]
\[ E = 0 : \text{circle} \]
\[ 0 < E < 1 : \text{ellipse} \]
\[ E = 1 : \text{parabola} \]
\[ E > 1 : \text{hyperbola} \]

Phase space foliated with neutrally stable periodic orbits.