Bifurcations

Topology of phase space flow changes as control parameter is varied.

Example: \( \dot{x} = r - x^2 \)

Flow on a line:

- \( r > 0 \)
- \( r = 0 \)
- \( r < 0 \)

Bifurcation diagram

Unstable:

Stable:

Saddle-node bifurcation:
Two fixed points collide and annihilate each other.
Ghost dynamics of saddle-node bifurcation

Limit cycle \( r > 0 \) homoclinic orbit \( r = 0 \) saddle node heteroclinic orbit \( r < 0 \)

Bifurcation oscillator stalled

\[ \dot{\theta} = \omega - \alpha \sin \theta \]

- Electronics:
  - Phase-locked loops
- Biology:
  - Oscillating neurons, fireflies
- Condensed matter physics
  - Josephson junctions

\[ T = \int dt = \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{\omega^2 - \alpha^2 \sin^2 \theta}} \approx \frac{1}{\sqrt{\alpha}} \cdot r = \omega - \alpha \]

Bifurcation parameter

This is generic:

\[ \dot{x} = r + x^2 \]

\[ \text{Bifurcation} \Rightarrow \int_{-\infty}^{\infty} \frac{dt}{\sqrt{r + t^2}} = \frac{\pi}{2} \]
Transcritical Bifurcation

\[ x^\prime = r x - x^2. \]

Normal form

\[ r < 0 \quad r = 0 \quad r > 0 \]
Minimal Model of Solid-State Lasers

\[ \dot{n} = G_n N - kn \]

- Gain
- Loss
- Stimulated emission
- Stimulated relaxation
- Pumping
- Mirror
- Active system (nonlinear dynamics)

\[ \dot{N} = -G_n N - f N + p \]

\[ n = \text{number of photons} \]
\[ N = \text{number of excited atoms} \]

Active hole elimination:

\[ N \text{ relaxes and feeds back into } n : \]
\[ \dot{N} = 0, \Rightarrow \]
\[ N = \frac{p}{f + G_n} \approx \frac{p}{f} - \frac{pG_n}{f^2 n} \]
\[ \frac{\dot{n}}{n} = G_n \left( N_0 - \frac{f}{f^2 n} \right) - kn. \]

\[ \Rightarrow \text{transcritical bifurcation} \]
Example:

**Double-well potential**

(prototype of system with symmetries)

\[ V(x) = -\frac{1}{4} x^4 + \frac{1}{2} x^2. \]

\[ \dot{x} = x - x^3. \]  \[ T = \frac{1}{2} u. \]

\[ r < 0 \]

\[ r = 0 \]

\[ r > 0 \]

Critical slowing down

Spontaneous symmetry breaking

\[ \dot{x} = -x^3 \]

(Perturbations decay algebraically in time, not exponentially.)
Losing Magnetic

$S_i = \pm \frac{1}{2}, \; i = 1, \ldots, N$.

$m = \langle S_i \rangle \equiv \text{magnetization}$

$m = 0 : \text{para magnet}$

$m > 0 : \text{ferromagnet}$

Hamiltonian

$$H = \sum_{i,j} \mathcal{E} \cdot S_i \cdot S_j, \; \mathcal{E} \equiv \text{interaction strength}$$

Mean field approximation

$$H \approx \frac{1}{N} \sum_{i} \mathcal{E} \cdot S_i \cdot n \cdot m$$

N energy per spin

$n = \text{number of neighbors}$

Partition function

$$Z = \sum_{S_i = \pm 1} \exp \left( - \frac{H_i}{\mathcal{E}} \right)$$

$$\langle S_i \rangle = (\langle S_i \rangle^+) p_+ + (\langle S_i \rangle^-) p_- = \exp \left( - \frac{\mathcal{E} \cdot n \cdot m}{\mathcal{E}} \right) - \exp \left( + \frac{\mathcal{E} \cdot n \cdot m}{\mathcal{E}} \right) \frac{Z}{e^\mathcal{E} \cdot n \cdot m} = - \tanh \left( \frac{\mathcal{E} \cdot n \cdot m}{\mathcal{E}} \right)$$

Self-consistency

$$\langle S_i \rangle \approx \langle m \rangle \Rightarrow kT \cdot \tanh \left( \frac{1}{2} \mathcal{E} \cdot m \right) = \mathcal{E} \cdot m$$
Pitchfork bifurcation

Super-critical

\[ x = r x - x^3 \]

Stable

Unstable

Stable

Sub-critical

\[ x = r x + x^3 - x^5 \]

Stability non-linear to avoid blow-up.

\[ \Rightarrow \text{ Dangerous in engineering applications.} \]
Index Theory =
topology of vector fields
in plane plane

- for any closed curve $C$
define index $I_c$:

$$V(x, y) = (x, y)$$

$I_c$ = how often does $V$ rotate when we go along $C$?

$$= \frac{1}{2\pi} \int_C \tan^{-1} y \, dx$$

**Examples**

- $I_c$ depends continuously on $C$
given an $C$ does not pass through any fixed point
  through any fixed point,
  continuous, integer-valued

$$\Rightarrow I_c = 0 \text{ if } C \text{ does not encircle any fixed point.}$$

Proof: Shrink $C$ to zero.
Theorem:

For each isolated fixed point $x^*$

We can define well-defined sets $I_{x^*}$ (take any curve encircling only $x^*$).

If closed curve $C$ encircles isolated fixed points $x_1^*, \ldots, x_n^*$, then

$I_c = I_{x_1^*} \cdot \ldots \cdot I_{x_n^*}$.

Proof:

- Any closed orientable curves $I_c = +1$, hence must contain at least one fixed point.

More precisely

# nodes + # spirals + # centers

$- #$ saddles

\[ = +1 \]

(For surfaces of higher topological genus depends on homotopy class of $c$)
Bifurcations revisited

In: continuous value change of bifurcation parameter, hence constant.

Saddle node bifurcation

Stable + unstable → φ

\[ \begin{array}{c}
0 \\
1 \\
-1 \\
0
\end{array} \]

Transcritical bifurcation

\[ \begin{array}{ccc}
0 & + & 0 \\
+1 & +1 & +1 & +1
\end{array} \]

Pitchfork bifurcation

\[ \begin{array}{ccc}
0 & + & 0 \\
+1 & +1 & -1 & +1
\end{array} \]