1. Sensitive dependence on initial conditions, yet dynamics bounded domain.

formally: positive Lyapunov exponents.

\[ \frac{d(t)}{\Delta t} = x_1(t) - x_2(t) \sim \text{do exp } \lambda t. \]

How possible? Stretching in one direction, compression in the other.

2. Deterministic

3. aperiodic long-term behavior.
(no fixed points, periodic orbit, quasi-periodic orbit)

\[ \Rightarrow \text{deterministic chaos.} \]

- Attractor sets are often fractals.
- Example (Lorenz, 1963)

\[
\begin{align*}
\dot{x} &= \sigma(y-x) \quad \sigma = 10 \\
\dot{y} &= r x - y - x z \quad r = 28 \\
\dot{z} &= x y - b z \quad b = 8/3
\end{align*}
\]
\[ n = 1 \quad \Rightarrow \quad \text{only fixed points} \]
\[ n = 2 \quad \Rightarrow \quad \text{oscillations, no chaos} \]
\[ n = 3 \quad \Rightarrow \quad \text{chaos} \]
\[ n > 10^3 \quad \Rightarrow \quad \text{statistical mechanics} \]

- **Swinging Atwood machine**
  
- **Double pendulum**
1D. Maps. (discrete time)

\[ X_{n+1} = f(x_n) \]

**Logistic Map.**

\[ X_{n+1} = r X_n (1 - X_n) = f(x_n) \]

**Q:** fixed points?

**A:**

\[ x = 0 \] for \( r \leq 1 \).

\[ x = 0 > 0 \] for \( r > 1 \).

\[ f(x^*) = x^* \implies x^* = \frac{r - 1}{r} \]

**Q:** stability?

**A:** \( x^* \) stable \( \iff \) \( f'(x^*) > -1 \) \( \iff \) \( r < 3 \).

- \text{Start with } x_0 = x^* + \epsilon
- \text{cobweb construction}
- \text{in 'if' play role of Lyapunov exponent.}
Q: What happens for $r > 3$?
A: period-doubling bifurcation

$f^2(x) = x \implies \left(2x(1-x)[1- r x (1-x)]\right) = x. \implies$

factor out trivial solutions.

$x = 0$ and $x = \frac{r-2}{r}$

$p \cdot q = \frac{r+1 \pm \sqrt{(r-1)(r+1)}}{2r}$

@ $r = 3$: $p = q = x^*$
Q: What happens for large $r$?

A:

- period doubling bifurcation at $r_a$
- $\delta = \lim_{n \to \infty} \frac{r_n - r_{n-1}}{r_{n+1} - r_n} = 4.669$
- $r > r_a$: chaos (= aperiodic motion)
- $\Rightarrow$ period doubling route to chaos
- even for $r > r_a$, there are windows of periodic motion.
- periodic orbits appear in universal sequence
  \[1, 2, 2 \times 2, 6, 5, 3, 2 \times 3, \ldots\]
- for \( r \) slightly below the lower bound of periodic window one observes intermittency
- long periods of almost periodic motion
- bursts of chaotic behavior
- gain periods of periodic motion
- Metropolis: Universal seq. for all unimodal maps
  \[x_{n+1} = \frac{x_n}{f(x_n)}, f(0) = f(1) = 0.\]
- Tjesen-Bassin: \( S = 4.669 \) for doubling bifurcation for all unimodal maps.
Experimental test

- Convection rolls, (Libchab)

\[ \text{cold} \quad \Rightarrow \quad \text{hot} \]

- diodes, Josephson junctions,...
Logistic map continued:

The period doubling route to chaos of Renormalization Theory for pedestrians.

Logistic map:

\[ X_{n+1} = r X_n (1 - X_n), \text{ so control parameter} \]

\[ r = r_k : \text{birth of } 2^k \text{-cycle.} \]

Let \( r_0 = r_k - r_n. \)
Want to show: \( r_n \) converges to \( r_m - r_n \) from some \( r_k. \)

**Task:** Compute \( r_n \) iteratively.

**Start:** We know \( r_0 = 0. \)

\( \Rightarrow \) Normal form of period-doubling bifurcation

\[ y = x - x^3, \quad x^3 = f(x^3) \]

\[ y_{n+1} = -(1 + \mu) y_n + \alpha y_n^2 + \ldots \]

\[ \Rightarrow \quad \nu \text{ for } a = 1 \]

\[ y_{n+1} = -(1 + \mu) y_n + y_n^2 = 0 (y_n) \]

\( \Rightarrow \) find fixed point of 2nd iterate \( y_{n+2} = y \)

\[ \mu^2 + 4\mu \]

\[ \left\{ \begin{array}{l} \mu^2 + 4\mu \\ y^2 = 0 \\ y \end{array} \right. \]

\[ x = \frac{\mu + \sqrt{\mu^2 + 4\mu}}{2} \]

\[ y = 0 \]

\[ \mu \]
What happens if we add a perturbation to $p$?

\[ p : \quad p = g^2(p). \]

\[ p + \varepsilon_n : \quad p + \varepsilon_{n+1} = g^2(p + \varepsilon_n) \]
\[ = p + (1 - \mu + \mu^2) \varepsilon_n + C \varepsilon_n^2 \]
\[ C = 4 \mu + \mu^2 - 3 \sqrt{\mu^2 + 4 \mu}. \]

Set \( \tilde{\varepsilon}_n = C \varepsilon_n \)

\[ \tilde{\varepsilon}_n = (1 - \mu + \mu^2) \tilde{\varepsilon}_n + \tilde{\varepsilon}_n^2 \]
\[ = - (1 + \tilde{\mu}) \tilde{\varepsilon}_n + \tilde{\varepsilon}_n^2 \]
\[ = \tilde{\varepsilon}_n (\tilde{\varepsilon}_n) \quad \text{with} \quad \tilde{\mu} = \mu^2 + 4 \mu - 2. \]

We found normal form again, but with renormalized $\mu$.

\[ \Rightarrow \text{find fixed point of 2nd iterate} \quad (\tilde{\varepsilon}_{n+2} = \tilde{\varepsilon}_n) \]
\[ \Rightarrow \bar{\tilde{\varepsilon}_n} = \tilde{\mu} + \frac{\tilde{\mu}^2 + 4 \tilde{\mu}}{2} \]
Note: The following are equivalent events.

(i) Birth of $2^k$ cycle of $\overline{g}$ at $\tilde{F}_{k-1}$
(ii) Birth of $2^{k-1}$ cycle of $g^2$ at $\tilde{F}_{k-1}$
(iii) Birth of $2^{k-2}$ cycle of $g^4$ at $\tilde{F}_{k-2}$

We have

$$\tilde{F}_{k-1} = \mu_k^2 + 4 \mu_k - 2 = \ell(\mu_k).$$

$$\tilde{F}_{k-2} = \ell(\ell(\mu_k)).$$

Now, we can work backward:

$0 = \tilde{F}_1 = \ell(\mu_k) \Rightarrow \mu_k = -2 + \sqrt{6}$

$0 = \tilde{F}_0 = \ell(\mu_k) = \ell(\ell(\mu_k)) \Rightarrow \mu_k = \ell(\mu_k)$

Fixed point $\mu_k = \ell(\mu_k)$

$\mu_k \approx 0.56$. (true 0.57)
Next, $\mu^+ - \mu^-$ decay as explained:

$$S = \lim_{k \to \infty} \frac{\mu^+ - \mu^-}{\mu^+ - \mu^-} \quad \text{L'Hopital}$$

$$\frac{d\mu^-}{d\mu^+} \bigg|_{\mu^+ = \mu^-} = \frac{2\mu^+ + 4}{\times 5.12 \ (true \ 4.67)} \quad 10\% \ offset.$$
Renormalization for pedestrians.

\[ \mu_n = r e^{-n} : \text{birth of } 2^k \text{-cycle.} \]

Q: Can we compute \( \mu_n \)?

A: We know \( \mu_0 = 0 \).

Let \( y = x - x^2 \):

\[ y_{n+1} = -(1+\mu) y_n + a y_n^2 + \ldots \]

Wlog. \( a = 1 \).

\[(*) \quad y_{n+1} = -(1+\mu) y_n + y_n^2, \]

\[ p, q = \frac{\mu \pm \sqrt{\mu^2 + 4\mu}}{2} \]

\[ p + q y_{n+1} = \mu^2 (p + q y_n) \]

\[ = p + (1-4\mu - \mu^2) y_n^2 + C y_n^2 \]

\[ C = 4\mu + \mu^2 - 3\sqrt{\mu^2 + 4\mu} \]

Set \( \tilde{y}_n = C y_n \),

\[(***) \quad \tilde{y}_{n+1} = (1-4\mu - \mu^2) \tilde{y}_n + \tilde{y}_n^2 \]

\[ = -(1+\mu) \tilde{y}_n + \tilde{y}_n^2 \text{ with } \tilde{y}_0 = \mu^2 + \sqrt{\mu^2 + 4\mu} \]
birth of $2^2=4$ cycle:

\[ \tilde{\mu} = 0 \implies \mu_1 = \mu_2^2 + 4 \mu_2 - 2 \]

\[ \implies \mu_2 = -2 + \sqrt{16}. \]

Similarly,

\[ \mu_{k-1} = \mu_k^2 + 4 \frac{\mu_k}{\mu_k} - 2. \]

fixed point $\mu_{*0} \approx 0.56$ (true 0.53).

\[ g = \lim_{k \to \infty} \frac{\mu_{k-1} - \mu_+}{\mu_0 - \mu_+} \text{ L'Hopital } \]

\[ = \frac{d\mu_{*k}}{d\mu_+} \bigg|_{\mu = \mu_+} \]

\[ = 2 \mu_+ + 4 \approx 5.12 \text{ (true 4.67).} \]
Introduction to Fractals: The Cantor Set.

Consider the Koch map.

\[ f(x) = \begin{cases} 
\frac{1}{3}x & \text{if } x \in [0, \frac{1}{2}] \\
\frac{2}{3} + \frac{1}{3}x & \text{if } x \in \left(\frac{1}{2}, \frac{2}{3}\right) \\
\frac{1}{3}x & \text{if } x \in \left(\frac{2}{3}, 1\right)
\end{cases} \]

Q: Is there an attractor set?
A: After N iterations,

\[ f^n(x) = 0, 0220202020202020202... \]

Symbolic dynamics:

- N digits
- Only 0 or 2
- In base 3
- N shifted to the right.

Each random seq. of finite length will appear with finite probability.

Each \( x \in [0,1] \) that can be written with only digit 0 or 2 in base 3 will be approximated to arbitrary precision.

Attractor set = Cantor Set.
Geometric construction of Cantor set

Step 0
Step 1
Step 2
Step 3

Fractal dimension of a set $B \subseteq \mathbb{R}^n$

- cover $B$ with small circles/spheres of radii $r_i < \delta$.

- sum $d$-dimensional volumes

$w_d \sum r_i^d$, $w_d = \frac{d^{1/2}}{\Gamma(1+d/2)}$

$w_0 = 1$, $w_1 = 2$, $w_2 = \frac{4}{3}$, $w_3 = \frac{4}{5}$, ...

- take minimum over all possible covers. (and let $\delta \to 0$):

$H^d(B) = \lim \inf_{\delta \to 0} \left\{ \sum w_i \geq r_i \right\}$

- generalization of box-counting length.

$D = \text{Hausdorff - Besicovitch - dim.}$
Q: What is the fractal dimension of the Cantor set?

\[ C = \text{Cleft} \cup \text{Cright} \]

\[ H^2(C) = H^2(\text{Cleft}) + H^2(\text{Cright}) \]
\[ = 2 \cdot H^2(\text{Cleft}) \]

But also
\[ H^2(\text{Cleft}) = \left(\frac{1}{3}\right)^2 \cdot H^2(C) \]
\[ \Rightarrow 2 = 3^D \]
\[ \text{if } 0 < H^0(C) < \infty \]
\[ \Rightarrow D = \frac{\ln 2}{\ln 3} < 1. \]

- Deterministic realization: Barnsley's map.

Fractal dimension of exp. data.
- Cover with cubes of size \( \varepsilon \).
  \[ D_f = \lim_{\varepsilon \to 0} \frac{\ln N(\varepsilon)}{\ln(1/\varepsilon)} \]
- Often \( D_0 = D_{\text{Hausdorff}} \).
Tight synchronization and Arnold tongues.

Example \( \dot{\phi}_1 = \omega_1 \)
\( \dot{\phi}_2 = \omega_2 \)

Dynamics on torus

No coupling yet

\[ \frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q} \Rightarrow \text{phase-locked} \]
\[ \dot{\phi}_1 = n \cdot \dot{\phi}_2 \]

Rational case:

- \[ \frac{\omega_1}{\omega_2} = \frac{n}{m} \in \mathbb{Q} \Rightarrow \text{phase-locked} \]

Irrational case:

- Quasi-periodicity

= finite number of incommensurable freq.

Q: What happens for oscillators coupling

\[ \dot{\phi}_i = \omega_i - \frac{\lambda}{2} \sin (\phi_i - \phi_j) \]

\[ \dot{s} = \omega_1 - \omega_2 - \lambda \sin s, \quad s = \phi_i - \phi_j \]
This was proven mathematically for the discrete-time Adler map:

$$S_{n+1} = S_n + \frac{\Delta \omega T}{w} - \sqrt{1 - \sin^2 S_n} \mod 2\pi$$

$$\equiv \text{Circle map (Arnold 1965)}.$$ 

- Whether phase-locking occurs for given irrational $w$ depends on how well $w$ can be approximated by rationals $\frac{p}{q}$ in Number Theory, e.g., the Golden section $\frac{\sqrt{5} - 1}{2}$ or word $\frac{2}{7}$.

- $K = -\lambda T = 1$: phase-locking occurs in the Lebesgue measure 0, yet some quasi-periodic orbits remain.
Devil's stair case

\[ \lim_{n \to \infty} \frac{S_n - S_0}{m} \]

\[ W = a e^T \]

Continuous function which takes in finite small jumps.