Topological Speed Limits to Network Synchronization

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(Received 19 June 2003; published 18 February 2004)

We study collective synchronization of pulse-coupled oscillators interacting on asymmetric random networks. We demonstrate that random matrix theory can be used to accurately predict the speed of synchronization in such networks in dependence on the dynamical and network parameters. Furthermore, we show that the speed of synchronization is limited by the network connectivity and remains finite, even if the coupling strength becomes infinite. In addition, our results indicate that synchrony is robust under structural perturbations of the network dynamics.

DOI: 10.1103/PhysRevLett.92.074101 PACS numbers: 05.45.Xt, 87.10.+e, 89.20.–a, 89.75.–k

Complex networks have attracted considerable research interest in the recent past [1]. Whereas most studies have focused on the *structures* of various systems such as the World Wide Web, email networks, genetic networks, and biological neural networks [1,2], an equally important task is to understand the *collective dynamics* on such networks. Here, the question arises: How is the dynamics on a complex network influenced by its structure [2]?

Synchronization appears to be one of the simplest types of collective dynamics among coupled dynamical systems [3]. It occurs ubiquitously in artificial as well as natural networks as different as Josephson junction arrays [4] and biological neural networks [5]. To understand the dynamics of such networks, theoretical studies have emphasized systems consisting of simple units such as phase- and pulse-coupled limit-cycle oscillators [6,7]. Yet, although real-world networks often possess a complex connectivity structure, most studies of synchronization of coupled oscillators either are restricted to networks of globally coupled units and simple regular networks or apply some mean field limit [6,7]. Although exact results on synchronization in networks with a general structure have been obtained recently [8–10], it is still not well understood how the structure of a complex network affects dynamical features of synchronization.

In this Letter, we study the collective synchronization of pulse-coupled oscillators interacting on asymmetric random networks. We find that the speed of synchronization is restricted by the network connectivity and remains finite, even if the coupling strength becomes infinite. No such speed limit exists, however, in large networks of globally coupled units. More generally, we show that the theory of random matrices can be used to successfully predict the speed of synchronization as a function of dynamical and network parameters. In addition, our results indicate that synchrony occurs robustly, i.e., persists under structural perturbation of the network dynamics. Random matrix theory has previously been applied to various physical systems that exhibit certain symmetries, e.g., time-reversal symmetry, but an otherwise unknown structure. For instance, correlations of energy levels in nuclear physics and quantum mechanical properties of classically chaotic systems have been successfully predicted (see Ref. [11] for a recent review). Our results demonstrate that random matrix theory also is an appropriate tool for analyzing synchronization in random networks of dynamical units.

We consider asymmetric random networks of N pulse-coupled oscillators [12]. The sets $\mathrm{Pre}(i)$ of presynaptic oscillators connected to (and thus affecting the dynamics of) oscillator i specify the structure of such a network. For each oscillator i, the $k_i := |\mathrm{Pre}(i)|$ presynaptic oscillators are drawn randomly from the uniform distribution among all other oscillators $\{1, \ldots, N\}\backslash\{i\}$.

The state of each oscillator i at time t is specified by a single phaselike variable $\phi_i(t)$. In the absence of interactions, its dynamics is given by

$$d\phi_i/dt = 1. \tag{1}$$

When oscillator i reaches a threshold, $\phi_i(t) = 1$, its phase is reset to zero, $\phi_i(t^+) = 0$, and the oscillator is said to fire. A pulse is sent to all postsynaptic oscillators $j \in \mathrm{Post}(i)$ which receive this signal after a delay time τ. The incoming signal induces a phase jump

$$\phi_j((t+\tau)^+) := U^{-1}[U(\phi_j(t+\tau)) + \varepsilon_{ji}], \tag{2}$$

which depends on the instantaneous phase $\phi_j(t+\tau)$ of the postsynaptic oscillator and the coupling strength ε_{ji} which we take to be inhibitory (phase retarding), $\varepsilon_{ji} \leq 0$. The phase dependence is determined by a twice continuously differentiable "potential" function $U(\phi)$ that is assumed to be strictly increasing, $U'(\phi) > 0$, concave (down), $U''(\phi) < 0$, and normalized such that $U(0) = 0$, $U(1) = 1$.

This phase dynamics is equivalent (cf. also [12]) to ordinary differential equations

$$dV_i/dt' = f(V_i) + I_i(t'), \tag{3}$$

where $I_i(t') = \sum_{j,m} \varepsilon_{ij}\delta(t' - t'_{j,m} - \tau')$ is a sum of delayed δ currents induced by the oscillators $j \in \mathrm{Pre}(i)$ sending their mth pulse at time $t'_{j,m}$. A pulse is sent by oscillator j whenever a threshold is crossed, $V_j(t'_{j,m}) \geq 1$, leading to an instantaneous reset of that oscillator, $V_j(t'^{+}_{j,m}) = 0$. The

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positive function $f(V) > 0$ yields a free ($I_i \equiv 0$) solution $V(t') = V(t' + T_0)$ of intrinsic period T_0. The above function U is related to this solution via

$$U(\phi) := V(\phi T_0), \tag{4}$$

defining a natural phase ϕ by rescaling the time axis, $t = t'/T_0$ and $\tau = \tau'/T_0$.

We focus on the specific form $U(\phi) = U_{\rm IF}(\phi) = I(1 - e^{-T_{\rm IF}\phi})$ that represents the integrate-and-fire oscillator defined by $f(V) = I - V$. Here $I > 1$ is a constant external input and $T_{\rm IF} = \ln[I/(I-1)]$ the intrinsic period of an oscillator. Other forms of $U(\phi)$ give qualitatively similar results. In such a network the synchronous state, $\phi_i(t) = \phi_0(t)$ for all i, exists if the coupling strengths are normalized such that $\sum_{j\in{\rm Pre}(i)}\varepsilon_{ij} = \varepsilon$. Its period is given by $T = \tau + 1 - U^{-1}[U(\tau) + \varepsilon]$.

In numerical simulations of the network dynamics, we find that the synchronous state is always stable, independent of the parameters (cf. [9]). A sufficiently small perturbation $\boldsymbol{\delta}(0) \equiv \boldsymbol{\delta} = (\delta_1, \ldots, \delta_N)^{\mathsf T}$ of the phases, defined by $\phi_i(0) = \phi_0(0) + \delta_i$ asymptotically decays exponentially with time. Thus, denoting $\boldsymbol{\delta}'(t) := \boldsymbol{\delta}(t) - \lim_{s\to\infty}\boldsymbol{\delta}(s)$, the distance $\Delta(n) := \max_i|\delta_i'(nT)|/\max_i|\delta_i'(0)|$ from the synchronous state behaves as

$$\Delta(n) \sim \exp(-n/\tau_{\rm syn}), \tag{5}$$

defining a synchronization time $\tau_{\rm syn}$ in units of the collective period T. The speed of synchronization $\tau_{\rm syn}^{-1}$ strongly depends on the parameters. For instance, as might be expected, synchronization is faster for stronger coupling. Surprisingly, however, we find that synchronization cannot be faster than an upper bound even if the coupling strength becomes arbitrarily large (cf. Fig. 1).

To understand how the speed of synchronization depends on the dynamical and network parameters, we analyze the linear stability of the synchronous state.

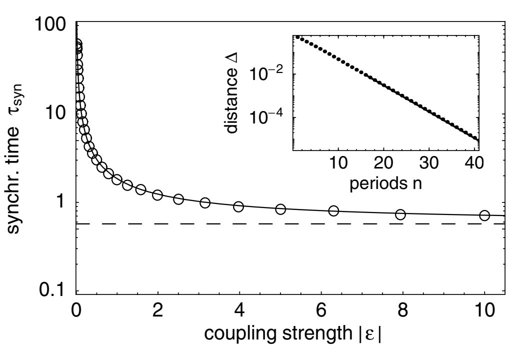

FIG. 1. Asymptotic synchronization time in a random network [$N = 1024$, $k_i \equiv k = 32$, $I = 1.1$, $\tau = 0.05$, $\varepsilon_{ij} = \varepsilon/k$ for $j \in {\rm Pre}(i)$] . The inset shows the distance Δ of a perturbation $\boldsymbol{\delta}$ from the synchronous state versus the number of periods n ($\varepsilon = -0.4$). Its slope yields the synchronization time $\tau_{\rm syn}$ shown in the main panel as a function of coupling strength $|\varepsilon|$. Simulation data (○), theoretical prediction (solid line) derived in this Letter, and its infinite coupling strength asymptote (dashed line).

Following [9] we obtain a nonlinear stroboscopic map $\boldsymbol{\delta}(T) = F(\boldsymbol{\delta})$ for the perturbations giving the perturbation $\boldsymbol{\delta}(T)$ after one period T in terms of the initial perturbation $\boldsymbol{\delta}$. Linearizing this map yields

$$\boldsymbol{\delta}(T) \doteq A\boldsymbol{\delta}, \tag{6}$$

defining the stability operator A that depends on $U(\phi)$ and the ε_{ij}. For $U(\phi) = U_{\rm IF}(\phi)$ and coupling strengths $\varepsilon_{ij} = \varepsilon/k_i$ for $j \in {\rm Pre}(i)$ the matrix elements reduce to [13]

$$A_{ij} = \begin{cases} \frac{1-A_0}{k_i}, & \text{if } j \in {\rm Pre}(i), \\ A_0, & \text{if } j = i, \\ 0, & \text{if } j \notin {\rm Pre}(i) \cup \{i\}, \end{cases} \tag{7}$$

where

$$A_0 = \frac{Ie^{-\tau T_{\rm IF}}}{Ie^{-\tau T_{\rm IF}} - \varepsilon} > 0. \tag{8}$$

The matrix A is row stochastic, i.e., $\sum_j A_{ij} = 1$ for all i and $A_{ij} \geq 0$ for all i and j. Thus A has one trivial eigenvalue $\lambda_1 = 1$ associated with the eigenvector $\boldsymbol{v}_1 = (1, 1, \ldots, 1)^{\mathsf T}$ representing a uniform phase shift and thus reflecting time-translation invariance. Furthermore, the Gershgorin theorem [15] implies that all eigenvalues are located inside a disk of radius $r_{\rm G} = 1 - A_0$ centered at A_0, such that, in particular, $|\lambda_i| \leq 1$ and the synchronous state is (at least neutrally) stable. For simplicity, we consider networks of homogenous random connectivity, $k_i = k$ for all i, in the following.

We numerically determined the eigenvalues of different stability matrices A for various network sizes $N \in \{2^6, \ldots, 2^{14}\}$, in-degrees $k \in \{2, \ldots, 2^8\}$, and dynamical parameters ε, τ, and I such that A_0 covers the entire accessible range $A_0 \in (0, 1)$. In general, we find that for sufficiently large k and N the nontrivial eigenvalues resemble a disk in the complex plane that is centered at about A_0 and has a radius r that is smaller than the upper bound given by the Gershgorin theorem, $r < r_{\rm G} = 1 - A_0$. Examples are shown in Fig. 2.

This eigenvalue distribution is reminiscent of the "circle law" of random matrix theory [16]: Gaussian asymmetric random matrices, having a distribution of matrix elements

$$p_{\rm Gauss}(J_{ij}) = N^{1/2}(2\pi r^2)^{-1/2}\exp\left(-\frac{NJ_{ij}^2}{2r^2}\right) \tag{9}$$

with independent J_{ij} and J_{ji}, also exhibit eigenvalue distributions

$$\rho^{\rm a}_{\rm Gauss}(\lambda) = \begin{cases} (\pi r^2)^{-1}, & \text{if } |\lambda| \leq r, \\ 0, & \text{otherwise}, \end{cases} \tag{10}$$

for $N \to \infty$ that are uniform in a disk in the complex plane [16]. The radius r of the disk is given by

$$r = N^{1/2}\sigma, \tag{11}$$

where $\sigma^2 = \langle J_{ij}^2\rangle$ is the variance of the matrix elements.

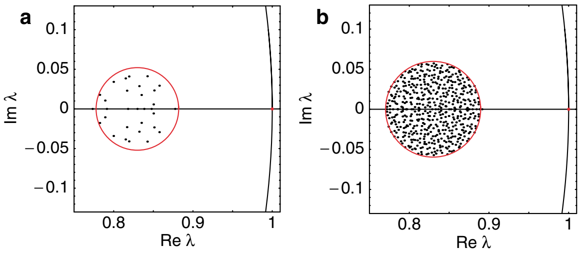

FIG. 2 (color online). Distribution of eigenvalues λ_i of two stability matrices A in the complex plane ($I = 1.1$, $\varepsilon = -0.2$, $\tau = 0.05 \Rightarrow A_0 \approx 0.83$; $k = 8$) for networks of (a) $N = 32$, (b) $N = 512$ oscillators. For large networks, the nontrivial eigenvalues seem to be distributed uniformly on a disk in the complex plane. The prediction from random matrix theory [Eq. (14)] is indicated by a circle. The arc through the trivial eigenvalue $\lambda_1 = 1$ is a sector of the unit circle.

Interestingly, we find that the radii of the eigenvalue distributions of the above stability matrices (7) agree well with the radii obtained from Eq. (11) if $\langle J_{ij}^2 \rangle$ is replaced by the variance of the elements of the stability matrices shifted such that they also exhibit a zero average eigenvalue. To directly compare the eigenvalues of the stability matrices, which have average eigenvalue $[\lambda_i] := \frac{1}{N}\sum_{i=1}^{N} \lambda_i = A_0$ to those of the Gaussian ensemble, we transform $A'_{ij} = A_{ij} - \delta_{ij} A_0$ shifting the average eigenvalues to $[\lambda'_i] = 0$. Here δ_{ij} denotes the Kronecker delta, $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ if $i \neq j$. For the variance of A' we obtain

$$\sigma_{A'}^2 = [A'^2_{ij}] - [A'_{ij}]^2 \tag{12}$$

$$= \frac{1}{N}\left(\sum_{\substack{j=1\\ j\neq i}}^{N} A_{ij}^2 - \frac{(1-A_0)^2}{N}\right). \tag{13}$$

For identical nonzero coupling strengths, the off-diagonal sum is exactly equal to $\sum_{j=1, j\neq i}^{N} A_{ij}^2 = (1 - A_0)^2/k$ such that, using (11), we obtain the random matrix theory prediction

$$r_{\mathrm{RMT}} = N^{1/2}\sigma_{A'} = (1 - A_0)\left(\frac{1}{k} - \frac{1}{N}\right)^{1/2}. \tag{14}$$

for the radius r of the disk of eigenvalues of the stability matrices A [17].

We verified this scaling law (14) for various dynamical parameters A_0 (determined by different I, ε, and τ), network sizes N, and in-degrees k and found excellent agreement with numerically determined eigenvalue distributions; see, e.g., Fig. 2. To quantify the accuracy of the prediction (14), we numerically estimated the radius of the distribution of the nontrivial eigenvalues of A for various N, k as well as A_0. Results from two different estimators are shown in Fig. 3. The real part estimator $r_{\mathrm{Re}} := \frac{1}{2}[\max_{i\neq 1}\mathrm{Re}(\lambda_i) - \min_{i\neq 1}\mathrm{Re}(\lambda_i)]$ estimates the radius from the maximum spread of eigenvalues parallel to the real axis. Typically, r_{Re} should give an estimate that is slightly inaccurate because it is based on two eigenvalues only. This is circumvented by the average estimator $r_{\mathrm{av}} := \frac{3}{2}\frac{1}{N-1}\sum_{i=2}^{N} |\lambda_i - [A_0 - (1 - A_0)N^{-1}]|$ that estimates the radius r of a circle from the average distance $\langle d \rangle$ of eigenvalues from its center, because $\langle d \rangle = \int_0^{2\pi}\int_0^r r'^2 \rho(r') dr' d\varphi = \frac{2}{3} r$ if we assume a uniform density $\rho(r')$ according to (10). Here we take the center of the disk to be the average $\langle \lambda_i \rangle_{i\geq 2} = A_0 - (1 - A_0)N^{-1} + \mathcal{O}(N^{-2})$ of the nontrivial eigenvalues. Varying k at fixed N as well as N at fixed k yields excellent agreement between the numerical data and the theoretical predictions for sufficiently large N and k (Fig. 3). Varying the coupling strength $|\varepsilon|$ and thus A_0 yields equally good agreement (cf. Fig. 1).

The radius (14) implies a prediction for the synchronization time [see (5)]

$$\tau_{\mathrm{syn}} = -1/\ln(A_0 + r_{\mathrm{RMT}}), \tag{15}$$

in terms of the (in modulus) largest nontrivial eigenvalue $\lambda_{\mathrm{m}} \approx A_0 + r_{\mathrm{RMT}}$. With increasing coupling strength $|\varepsilon|$, the synchronization time decreases. However, the speed of synchronization τ_{syn}^{-1} is bounded by a finite speed for arbitrary large $|\varepsilon|$: Even if $|\varepsilon| \gg 1$ and thus $A_0 \ll 1$, the largest nontrivial eigenvalue asymptotically becomes $\lambda_{\mathrm{m}} \approx k^{-1/2}$ for large N. Thus, the shortest synchronization time

$$\tau_{\mathrm{syn}}^{|\varepsilon|\to\infty} = \frac{2}{\ln k} \tag{16}$$

is limited by the network connectivity (cf. the asymptote in Fig. 1). This means that even for arbitrary strong interactions, the speed of synchronization stays finite. Furthermore, at fixed k, the synchronization time also cannot exceed a certain maximum, even if the network size N becomes extremely large (cf. Fig. 3). This bound

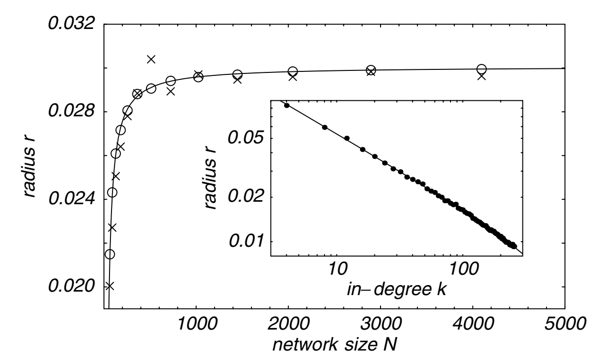

FIG. 3. Scaling of the radius r of the disk of nontrivial eigenvalues. Main panel displays the radius r as a function of network size N for fixed $k = 32$. Symbols display r_{Re} ($\times$) and r_{av} ($\bigcirc$). Inset displays r as a function of k for fixed $N = 1024$. Dots display numerical data of r_{av}. In the main panel and the inset, lines are the theoretical estimate r_{RMT} [Eq. (14)]. Other parameters are as in Fig. 2.

$\tau_{\rm syn}^{N\to\infty}$ is determined by the asymptotic radius $r_\infty := \lim_{N\to\infty} r_{\rm RMT} = (1 - A_0)k^{-1/2}$. Moreover, because eigenvalues change continuously with a structural perturbation to the system's dynamics, the existence of a gap $g := 1 - (A_0 + r_\infty) > 0$ indicates that no eigenvalue crosses the unit circle for sufficiently small structural perturbations. Thus stable synchrony is not restricted to the specific model considered here, but should persist in systems obtained by structural perturbations of the dynamics.

The above results show that the distribution of eigenvalues of a *sparse* stability matrix with deterministic nonzero entries at certain random positions is well described by the eigenvalue distribution of the *Gaussian* ensemble, which consists of fully occupied matrices with purely random entries. This sparse-Gaussian coincidence for asymmetric matrices is similar to that of symmetric random matrices for large k: Gaussian symmetric matrices exhibit an eigenvalue distribution $\rho^{\rm s}_{\rm Gauss}(\lambda)$, the Wigner semicircle law [18]. Sparse symmetric matrices [19] exhibit an eigenvalue distribution $\rho^{\rm s}_{\rm sparse}(\lambda)$ that is different from the semicircle law but approaches it in the limit $k \to \infty$. For $k \gg 1$ the distribution of eigenvalues of sparse asymmetric random matrices $\rho^{\rm a}_{\rm sparse}$ appear to be well approximated by the eigenvalue distribution of Gaussian asymmetric matrices, $\rho^{\rm a}_{\rm sparse}(\lambda) \approx \rho^{\rm a}_{\rm Gauss}(\lambda)$. Our results indicate that this is true even for moderate $k \approx 10$.

Further investigations of eigenvalue distributions for small-world networks show that with decreasing randomness the speed of synchronization decreases (the second largest nontrivial eigenvalue increases) such that fully random networks synchronize faster than small-world networks, at least asymptotically.

In conclusion, we have derived accurate analytical predictions for the (asymptotic) speed of synchronization in asymmetric random networks of oscillators in dependence of the dynamical parameters ε, τ, I, as well as the network parameters N and k. Even the scaling with network size N, artificially introduced via the variance (13) of finite matrices, is accurately reproduced (see, e.g., Fig. 3). The analysis revealed that, as well as how the speed of synchronization is restricted by the network connectivity.

Possible lines for future applications of our random matrix theory approach may include synchronization phenomena of pulse- and phase-coupled units as well as of chaotic dynamical systems (cf. also [20]). The speed limits of synchronization predicted in this Letter are expected to occur in those systems, too. More generally, other equilibration processes and the dynamics in more structured topologies such as small-world networks may be analytically investigated using statistical spectral properties of the respective operators.

We thank S. Großkinsky, T. Kottos, P. Müller, H. Sompolinsky, M. Weigt, and A. Zippelius for useful discussions.

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