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Braess's paradox in oscillator networks, desynchronization and power outage

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Abstract. Robust synchronization is essential to ensure the stable operation of many complex networked systems such as electric power grids. Increasing energy demands and more strongly distributing power sources raise the question of where to add new connection lines to the already existing grid. Here we study how the addition of individual links impacts the emergence of synchrony in oscillator networks that model power grids on coarse scales. We reveal that adding new links may not only promote but also destroy synchrony and link this counter-intuitive phenomenon to Braess's paradox known for traffic networks. We analytically uncover its underlying mechanism in an elementary grid example, trace its origin to geometric frustration in phase oscillators, and show that it generically occurs across a wide range of systems. As an important consequence, upgrading the grid requires particular care when adding new connections because some may destabilize the synchronization of the grid—and thus induce power outages.

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Contents

1. Introduction

Synchronization constitutes one of the most prevalent collective dynamics in complex networked systems [1–6]. It underlies the function of systems across science and engineering, including pacemaker cells of the heart [7], communication networks [8] and electric power grids [9–12]. In particular, our supply with electric energy, a multi-billion Euro business, relies on a precisely locked partially synchronous state among power plants and consumers across the grid.

For electric power grids, the use of renewable and more distributed power sources steadily rises, thus requiring one to add new transmission lines or to increase existing lines' capacity in the near future, see [13]. Clearly, such lines have to reflect the future demand for transmissions, but at the same time global synchrony has to be maintained [14]. For power grids and more general synchronizing networks, this raises the key question 'Does adding new links support synchrony?'.

Here, we show how—when adding new links—not only the local transmission capacity but also equally global cycles are crucial for stable network synchrony. Specifically, we study the addition of single links in a class of oscillator networks modeling modern power grids on coarse scales. We show that whereas additional links stabilize synchronous operation *on average*, specific potentially new links decrease the total grid capacity and thus decrease or even destroy the locking on the grid. We link this phenomenon to Braess's paradox that was originally found in traffic networks [15], where closing a street may decrease the overall traffic congestion. For oscillator networks, it has been observed in simulations that removing links may increase synchronizability [16–18]. In electric circuits, Braess's paradox has been found in static (dc) settings of small systems [19, 20]. This notwithstanding, how Braess's paradox acts in large networks of oscillatory units is still not well understood and in oscillatory power grids it has not been detected so far.

We now systematically study Braess's paradox in oscillator networks and reveal the mechanisms underlying it. We identify its physical origin in the geometric frustration on small cycles in the network: phase differences along every cycle need to add up to multiples of 2π to make the phases of all oscillators well defined. Adding a link now creates new cycles and thus adds new consistency conditions for the stationary phase differences. Without those conditions satisfied, the synchronous state of 'normal operation' does not exist. Braess's paradox robustly emerges across complex network topologies and generalizes to a variety of networks of weakly coupled limit cycle oscillators. On a real supply grid, Braess's paradox may imply a costly power outage, possibly induced by a newly built connection line.

2. An oscillator network model for power grid operation

As a cornerstone example, consider networks of two-dimensional oscillators [12, 14], providing coarse-scale models of electric power grids. In this model, each of N rotating machines $j \in \{1, \ldots, N\}$ (representing, for instance, water turbines or electric motors) is characterized by the electric power P_j it generates ($P_j > 0$) or consumes ($P_j < 0$). The mechanical phase of each machine is written as $\theta_j(t) = \Omega t + \phi_j(t)$, where $\Omega = 2\pi \times 50$ or 60 Hz is the set frequency of the power grid. The approximate equations of motion for the difference phase $\phi_j(t)$ then follow directly from the conservation of energy (see appendix A for details). The power transmitted between two machines i and j is proportional to the sine of the relative phase $\sin(\phi_i - \phi_j)$ and the capacity of the respective transmission line K_{ij}. Taking the change in the kinetic energy of each machine and its mechanical dissipation into account yields

$$\frac{\mathrm{d}^2\phi_j}{\mathrm{d}t^2} = P_j - \alpha \frac{\mathrm{d}\phi_j}{\mathrm{d}t} + \sum_{i=1}^{N} K_{ij} \sin(\phi_i - \phi_j) \tag{1}$$

by equating the generated/consumed power to the transmitted, kinetic and dissipated power and assuming that $|\dot{\phi}_j| \ll \Omega$. A stable operation of the power grid is indicated by a stable phase-locked state of (1), with fixed phase differences $\phi_i - \phi_j$.

3. Identifying Braess's paradox in elementary power grids

First, consider an analytically solvable model as illustrated in figure 1. Four generators with $P_j = +P$ and four consumers with $P_j = -P$ are connected by transmission lines with a capacity K. As detailed below, a stable steady state exists for the original network structure (figure 1(a)) if the capacity exceeds a critical value

$$K > K_{\mathrm{c}} = P, \tag{2}$$

and the system rapidly relaxes to this phase-locked state of partial synchrony, see figure 1(d).

What happens if a transmission line is added or if the capacity of one line is increased? Naively, one expects that an additional capacity (cf figures 1(b) and (c)) increases the stability of the supply network. Paradoxically, the opposite holds—the oscillators do not phase lock (see figures 1(e) and (f)). Moreover, as shown below, no steady state exists for these parameters, because the critical connectivity K_{c} is larger than that for the original network.

An explanation for this counter-intuitive behavior is illustrated in figure 2. Panel (a) shows the phases of the oscillators ϕ_j and the flows $K_{ij} \sin(\phi_i - \phi_j)$ for the original network at the critical point. When the additional transmission line is added, the flows redistribute, see panel (b). More power is transmitted over the upper lines than over the lower ones. This induces

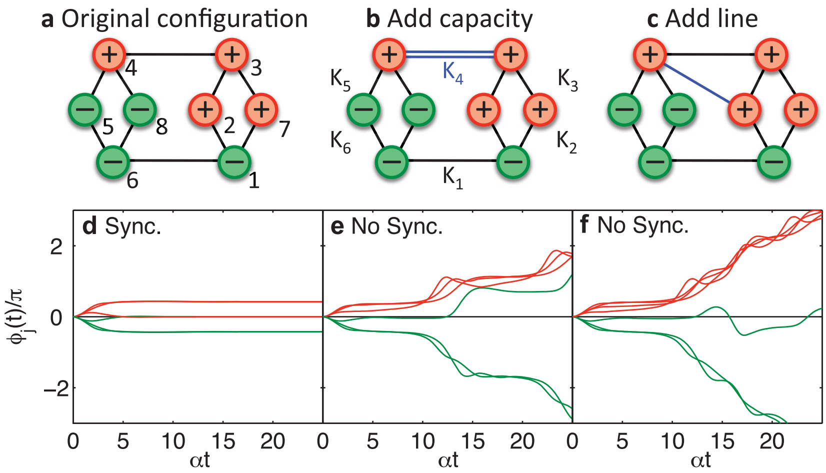

Figure 1. Elementary example of an oscillator network that exhibits Braess's paradox. (a)–(c) Topology of the network. The vertices generate/consume the power $P_j = \pm P$. The transmission lines have a capacity K. (d) The original network converges to a phase-locked state. When the capacity of one link is doubled (e) or when a new link is added to the network (f), the steady state ceases to exist and phase synchronization breaks down. Parameters are $K = 1.03\,P$, $\alpha = P$ and the initial conditions are $\phi_j = \dot{\phi}_j = 0$.

an overload in the connecting lines—the steady state is lost. A similar effect in traffic networks was first described and explained by Braess in a game theoretical framework [15, 21] and later confirmed experimentally and numerically in different systems, including internet routing [22, 23] or large street networks [24]. In the context of electric circuits, Braess's paradox has previously been demonstrated for static or dc models only [19, 20].

To quantitatively analyze the degree of synchrony of such networks, we consider the phase order parameter

$$r\,\mathrm{e}^{\mathrm{i}\psi} := \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\phi_j}, \tag{3}$$

setting $r := 0$ if no steady state exists. The magnitude of the order parameter is plotted in figures 2(c) and (d) as a function of K and the additional capacity ΔK, respectively. The example shows that adding capacity may reduce the synchronizability of the network by implying an increase of the critical coupling strength K_c. We remark that still, as soon as a steady state exists, the order parameter is increased.

4. Geometric frustration induces Braess's paradox

To better understand the origin of Braess's paradox in oscillator networks, we further analyze the steady state of the elementary example system in more detail. We focus on the scenario depicted in figure 1(b), where the capacity of the upper transmission line is gradually increased to $\tilde{K} = K + \Delta K$. Due to the symmetry of the problem, we have $\phi_7 = \phi_2$ and $\phi_8 = \phi_5$, such that we can treat two oscillators as one with doubled power and doubled connection strength,

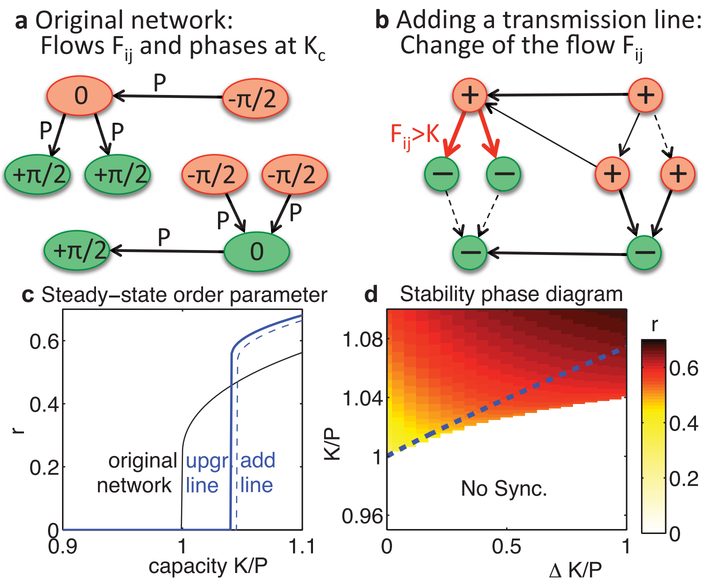

Figure 2. Quantitative analysis of Braess's paradox. (a) Flows $F_{ij} = K_{ij} \sin(\phi_i - \phi_j)$ and phases ϕ_j at the critical point $K_c = P$ for the original network. (b) When a new link is added, the transmission lines colored in red are overloaded and the steady state ceases to exist. (c) The steady-state order parameter r as a function of K for the original network (thin line), with one additional transmission line (see figure 1(c), dashed) and one transmission line upgraded by $\Delta K = K$ (see figure 1(b), thick line). (d) Stability phase diagram for a network with one upgraded line: order parameter r as a function of K and ΔK. No steady state exists in the white region. Dashed line: the theoretical linear approximation (8) predicts an increase of the critical coupling strength K_c with ΔK.

i.e. $P_2' = +2P$, $P_6' = -2P$ and $K_2 = K_3 = K_5 = K_6 = 2K$, whereas $K_1 = K$ and $K_4 = \tilde{K}$. For convenience we define the sines of the phases

$$S_j = \sin(\phi_j - \phi_{j-1}), \quad j \in \{1, \ldots, 6\}, \tag{4}$$

identifying $\phi_0 = \phi_6$. The S_j are bounded to the interval $S_j \in [-1, 1]$. The conditions for a steady state, $\ddot{\phi}_j = 0 = \dot{\phi}_j$ for all j, then reduce to a linear set of equations

$$0 = P_j + (K_{j+1} S_{j+1} - K_j S_j), \tag{5}$$

which yields an analytical solution for the phase-locked state (if it exists). Since one of the linear equations is redundant, the solutions span a one-dimensional space parametrized by a real number δ,

$$\vec{S} = \frac{P}{K}(\vec{S}_a - \delta \vec{S}_b) \tag{6}$$

with $\vec{S}_a = (1, 1, 0, -K/\tilde{K}, -1, 0)$ and $\vec{S}_b = (2, 1, 1, 2K/\tilde{K}, 1, 1)$. For the original network structure, where $\tilde{K} = K$, the condition $S_j \in [-1, 1]$ can be satisfied for all j if and only if $K > K_c = P$, defining the critical coupling for the existence of a steady state.

The analytic solution yields insights into a physical explanation for Braess's paradox in oscillator networks: suppose that the network is operated at the critical point $K = K_c$ and the capacity of the upper transmission line is increased to $\tilde{K} > K$. Then it is still possible to satisfy conditions (5) as well as $S_j \in [-1, 1]$ by setting $S_4 = -K/\tilde{K}$. However, there is another condition (not yet mentioned) that results from the cyclic path geometry in the network:

$$\sum_j (\phi_j - \phi_{j-1}) = \sum_j \arcsin(S_j) = 0\,(\mathrm{mod}\,2\pi), \tag{7}$$

i.e. the sum of all phase differences around a cycle must vanish such that all phases are well defined. For $K = K_c$, $\tilde{K} > K$, this condition is no longer satisfied. Despite the fact that all dynamical conditions (5) are satisfied, no steady state exists due to *geometric frustration*. In fact, the critical coupling strength is increased as shown in figures 2(c) and (d). Thus, geometric frustration limits the capability of the network to support a steady state. Related results for biological oscillators have been reported in [25, 26].

For small $\Delta K \ll K$ we analytically calculate the critical coupling strength K_c as follows. The synchronized state is lost when the link between oscillators 4 and 5 becomes overloaded. The critical state is thus characterized by $S_5 = -1$, for which we find $K_c = (1+\delta)P$ from equation (6). Similarly, we write $\tilde{K} = (1+\epsilon)P$ with a small parameter ϵ. Using again equation (6) we can calculate the phase differences $\phi_j - \phi_{j-1}$ to leading order in δ and ϵ. Substituting the result into condition (7) yields (see appendix B for details)

$$\epsilon = 2\delta + \tfrac{1}{2}[2\delta + (2+\sqrt{6})\sqrt{\delta}]^2 \tag{8}$$

and thus K_c as a function of ΔK. This function separates the synchronous and asynchronous phases, see figure 2(d).

5. Braess's paradox on complex network topologies

Braess's paradox is rooted in the geometric frustration of small cycles, which are generally present in most complex networks. Thus it occurs in many, but not all complex networks as the elementary cycles are typically overlapping such that the effects of geometric frustration are hard to predict.

As an important example, we consider the structure of the British high-voltage power transmission grid shown in figure 3, see [27]. In our study, we randomly chose 10 out of 120 nodes to be generators while the remaining ones are consumers. For $K = 13P_0$ the initial network relaxes to a phase-locked steady state as shown in panel (b) of the figure. If an inappropriate new link is added, global phase synchronization is lost due to Braess's paradox for the given coupling strength K. Instead, the power grid decomposes into two asynchronous fragments as shown in panel (c).

Furthermore, we analyze how synchrony is affected when a new link is added to the grid for 20 arbitrarily chosen positions. The addition of 9 out of the 20 potential new links leads to a significant change in the critical coupling strength K_c for the onset of synchronization. The common feature of these potential links is that transmission lines in their neighborhood are heavily loaded, see figure 4. Thus, adding new transmission lines has the strongest effect when they are built in regions where the existing lines are already heavily loaded. However, this effect is not necessarily positive—adding new lines may *increase* the load of existing lines and eventually induce Braess's paradox.

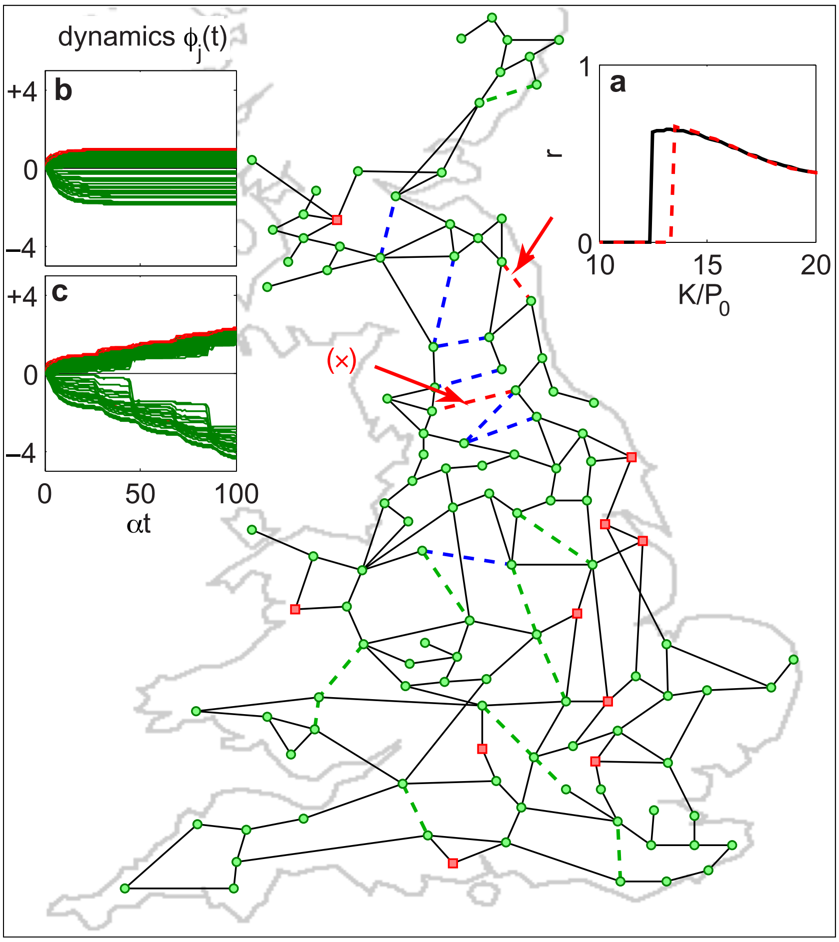

Figure 3. Possibility of Braess's paradox in a complex power grid. Synchronization is inhibited when the red dashed links are added to the network. Background: topology of the UK power grid, consisting of 120 nodes and 165 transmission lines (thin black lines) [27]. Ten nodes are randomly selected to be generators ($P_j = 11P_0$, □), and the others are consumers ($P_j = -P_0$, ○). To probe the effect of new links, the critical coupling K_c was calculated for the initial network and after the addition of a link at one out of 20 arbitrary positions (dashed lines). Two links increase K_c corresponding to Braess's paradox (red, marked by arrows), 7 decrease K_c (blue) and 11 leave K_c invariant (green). (a) Steady-state order parameter r as a function of the coupling strength K without (—) and with (- - -) the additional link marked by a cross (×). (b), (c) For $K = 13\,P_0$ and $\alpha = P_0$, the initial network converges to a phase-locked steady state (b), while this becomes impossible after the addition of the new link (×) (c). The initial condition is $\phi_j = \dot{\phi}_j = 0$.

6. Nonlocal impact and topology dependence

This negative effect is found for 2 out of the 20 potential links, i.e. their addition inhibits synchronization and leads to an increase of K_c. The nodes, where these two links are incident on,

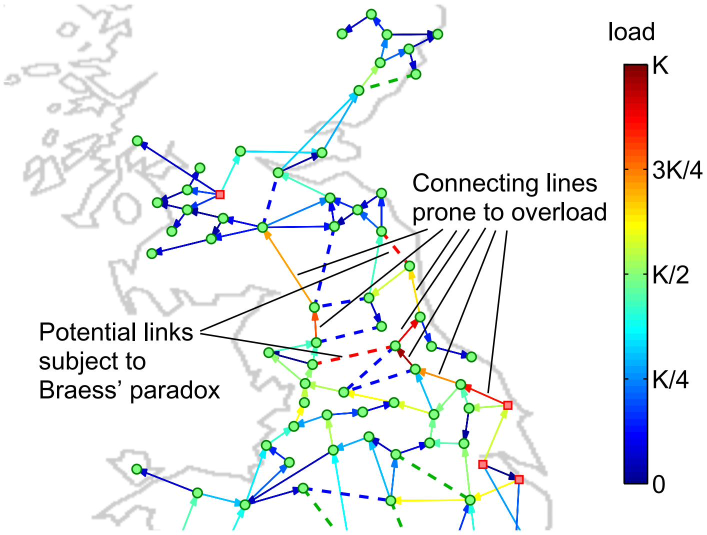

Figure 4. Nonlocal impact: the critical links in Braess's paradox. The figure illustrates the power flow in the UK power grid model studied in figure 3. The thin solid arrows indicate the load of each transmission line in the steady state in a colour code for $K = 13P_0$ before any new line is added. The dashed lines represent potential new links at arbitrarily chosen positions. Two links increase the critical coupling strength K_c, corresponding to Braess's paradox (red), 7 decrease K_c (blue), 11 leave K_c invariant (green). Braess's paradox can be caused by a *nonlocal collective effect*, i.e. an overload can occur at some distance from the added link.

receive electric energy through a series of heavily loaded transmission lines (marked in figure 4). If one critical potential link is put into operation, these connecting lines become overloaded, causing a loss of synchrony and thus a power outage. Having a closer look at the northern of the two potential links, we see that the most heavily loaded connecting line is not directly incident; rather it is located two links away to the south. Therefore, in order to predict the dynamics after structural changes in the power grid, one must not only analyze the immediate neighborhood but also take into account the possibility of strong *nonlocal collective effects*.

To systematically study how Braess's paradox depends on topological features, we simulate networks that interpolate between regular and random topology (cf [28, 29]). Starting with a square lattice, every link is rewired with probability q, i.e. removed and re-inserted at a different randomly chosen position. Half of the nodes are randomly chosen to be generators ($P_j = +P_0$) or consumers ($P_j = -P_0$). We then check for each link how its removal affects the synchronizability of the network, i.e how its removal changes the critical coupling strength K_c. Figure 5 shows the fraction of links whose removal increases or decreases K_c as a function of the network size N and the rewiring probability q.[4] A decreasing value of K_c shows that the respective link is preventing synchronization, i.e. it shows Braess's paradox. Figure 5(b) shows that this effect occurs mostly in regular networks (small q), where many small clusters exist which can suffer from geometric frustration. For large networks the removal of most links

[4] In these simulations $1/K$ is varied with a resolution of 0.01. Thus, smaller changes in K_c are not detected. See appendix C for details.

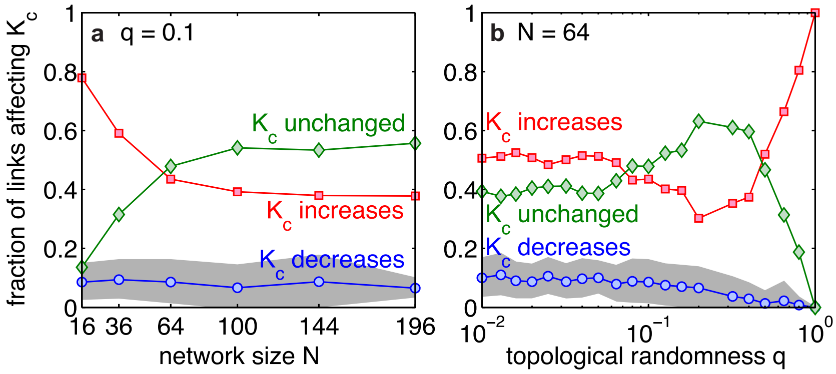

Figure 5. Braess's paradox systematically emerges across network topologies. The fraction of links in a network whose removal increases or decreases K_c, where the latter reflects Braess's paradox. Results are plotted (a) as a function of the number of nodes N and (b) the topological randomness q averaged over 100 network realizations. The shaded area shows the standard deviation for decreasing K_c.

does not affect K_c at all, as the impact of a single link decreases with the total number of links present. However, even in a network with $N = 196$ nodes, the removal of 6.5% of all links leads to a decrease of the critical coupling strength K_c. Thus Braess's paradox occurs for a significant fraction of links also in a large complex network.

7. Inverse percolation of complex networks

The significance of Braess's paradox becomes even clearer in the following inverse percolation processes. We consider a phase-locked oscillator network modeling the normal operation of a power grid, where single transmission lines randomly drop out of service one by one. For every network we calculate the order parameter r as a function of the coupling strength K and the critical coupling K_c for the existence of a steady state. As above, we consider an ensemble of networks interpolating between regular and random topology starting from a square lattice with 100 nodes and rewiring probability $q = 0.1$. Half of the nodes are randomly chosen to be generators or consumers, $P_j = \pm P_0$ (see appendix C for details).

The stability of such a network is summarized in a phase diagram as plotted in figure 6. When the phase with $r > 0$ is left due to the manipulation of a single link, then stability is lost *globally*. For real power grids, such an instability would trigger a major power outage (cf [27, 30–35]). The naive expectation is that the removal of links always destabilizes the power grid such that K_c increases monotonically with the number of missing links, but our previous reasoning has shown that this is not necessarily the case.

Averaging over 100 network realizations seems to confirm the naive expectation. However, Braess's paradox often occurs in concrete single realizations, see figure 6(b). In this case, the removal of the certain links leads to a decrease of K_c and thus stabilizes the power grid. Consequently, a global power outage can be caused not only by the removal, but also by the *addition* of single links. In fact, this paradoxical behavior happens quite regularly. Out of 1000 random realizations each, 981 networks exhibit Braess's paradox during the inverse percolation

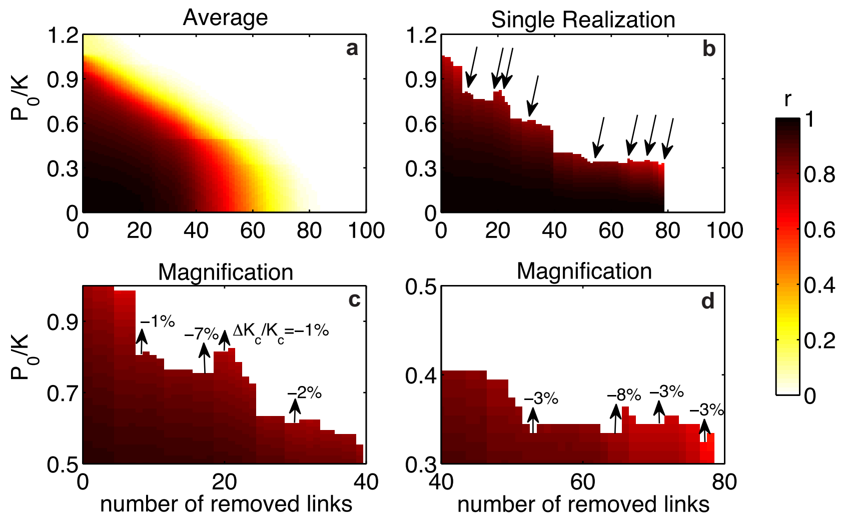

Figure 6. Inverse percolation subject to Braess's paradox. Shown is the order parameter r (color coded) as a function of $1/K$ and the number of removed links (a) averaged over 100 realizations and (b)–(d) for a single realization. The arrows mark the occurrence of Braess's paradox, where K_c decreases when a link is removed. The initial power grid consists of 50 generators ($P_j = +P_0$) and 50 consumers ($P_j = -P_0$) in a randomized square lattice with $q = 0.1$.

process. Notably, a substantial fraction of 8.5% of all links are subject to Braess's paradox (see table C.1).

Braess's paradox occurs slightly less frequently in heterogeneously powered networks. As an example, we choose 20% of the nodes to be generators ($P_j = +4P_0$) and the rest to be consumers ($P_j = -P_0$). In this case, we found that 902 out of 1000 random realizations and 4.8% of all links are subject to Braess's paradox.

8. Discussion

We have revealed and analyzed Braess's paradox for networks of coupled phase oscillators, in particular for a coarse-grained oscillator model of electric power grids: against the naive intuition, the addition of new connections in a network does *not* always facilitate the onset of phase-locked synchronization. An elementary model system was designed to analytically study this paradoxical behavior and to show the connection to geometric frustration. Direct numerical simulations have revealed that a substantial fraction of potential links induce this deleterious dynamical transition, with details systematically depending on the topology of the networks. We conclude that catastrophic failures of complex supply networks, in particular power grids, may not only be due to failure of single elements or links but even by the addition of *single* links. In particular, networks may destabilize due to a nonlocal overload.

The dynamical power grid model studied here complements statistical mean-field and flow models (see, e.g., [6, 27, 30, 36, 37]) and emphasizes the importance of single links in complex networks (cf [17, 35, 38, 39]). As a consequence of local dynamical equivalence to qualitatively similar models, our results equally hold for the power grid model (1), for the celebrated Kuramoto model [40–44] and for other phase oscillator models [2]. Thus, Braess's paradox is expected to be important across a variety of networks of weakly coupled limit cycle oscillators, also because it may have different consequences for different systems modeled as oscillator networks [2, 42].

For the future development of power grids, it would be extremely important to analyze the effects of local topological changes in advanced engineering models of real-world power grids[5]. Furthermore, future model studies must include the spatial as well as temporal heterogeneity of the power grid. For instance, the power generated by wind turbines is strongly fluctuating and its basic dynamics is actively controlled by power electronics. This notwithstanding, Braess's paradox is a general feature arising in supply networks of different types and we expect that it may play a crucial role in real power grids.

In the future, it will be of great scientific as well as economic interest to understand how these phenomena depend on the topologies of the underlying networks in detail and how they cause nonlocal overloads, cf [1, 6, 35, 45]. In particular, a badly designed electric power grid subject to Braess's paradox may cause enormous costs for new but counterproductive electric power lines that actually reduce grid performance and stability.

Acknowledgments

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Appendix A. The coupled oscillator model for power grids

Here we present a detailed derivation of the equations of motion for the power grid model, see equation (1). We consider a power grid model consisting of N rotating machines $j \in \{1, \ldots, N\}$ representing, for instance, water turbines or electric motors [12, 14]. Each machine is characterized by the electric power P_j it generates ($P_j > 0$) or consumes ($P_j < 0$). The state of each machine is determined by its mechanical phase angle $\theta_j(t)$ and its velocity $\mathrm{d}\theta_j/\mathrm{d}t$. During the regular operation, generators as well as consumers within the grid run with the same frequency $\Omega = 2\pi \times 50\,\mathrm{Hz}$ or $\Omega = 2\pi \times 60\,\mathrm{Hz}$, respectively. The phase of each element is then written as

$$\theta_j(t) = \Omega t + \phi_j(t), \tag{A.1}$$

where ϕ_j denotes the phase difference to the reference phase Ωt.

The equation of motion for all ϕ_j can now be obtained from energy conservation; that is, the generated or consumed energy $P_{\mathrm{source},j}$ of each machine must equal the energy sum given or taken from the grid plus the accumulated and dissipated energy. The dissipation power of each element is given by $P_{\mathrm{diss},j} = \kappa_j (\dot{\theta}_j)^2$, where κ_j is a friction coefficient. The kinetic energy of a rotating energy with a moment of inertia I_j is given by $E_{\mathrm{kin},j} = I_j\, \dot{\theta}_j^2/2$ such that the accumulated power is given by $P_{\mathrm{acc},j} = \mathrm{d}E_{\mathrm{kin},j}/\mathrm{d}t$. The power transmitted between two

[5] See, e.g., the power system simulation packages PSS/E (http://www.energy.siemens.com) and EUROSTAG (http://www.eurostag.be).

machines i and j is proportional to the sine of the relative phase $\sin(\theta_i - \theta_j)$ and the capacity of the respective transmission line $P_{\max,ij}$,

$$P_{\text{trans},ij} = P_{\max,ij}\sin(\theta_i - \theta_j). \tag{A.2}$$

If there is no transmission line between two machines, we have $P_{\max,ij} = 0$. The condition of energy conservation at each node j of the network now reads

$$P_{\text{source},j} = P_{\text{diss},j} + P_{\text{acc},j} + \sum_{i=1}^{N} P_{\text{trans},ij}. \tag{A.3}$$

Note that an energy flow between two elements is only possible if there is a phase difference between these two.

We now insert equation (A.1) to obtain the evolution equations for the phase difference ϕ_j. We can assume that phase changes are slow compared to the set frequency, $|\dot{\theta}_j| \ll \Omega$, such that terms containing $\dot{\phi}_j^2$ and $\dot{\phi}_j\ddot{\phi}_j$ can be neglected. Then one obtains

$$I_j\Omega\ddot{\phi}_j = P_{\text{source},j} - \kappa_j\Omega^2 - 2\kappa_j\Omega\dot{\phi}_j + \sum_{i=1}^{N} P_{\max,ij}\sin(\phi_i - \phi_j). \tag{A.4}$$

Note that in the equation only the phase difference ϕ_j to the reference phase Ωt appears. This shows that only the phase difference between the elements of the grid matters.

For the sake of simplicity we consider similar machines only such that the moment of inertia I_j and the friction coefficient κ_j are the same for all elements of the network. Defining $P_j := (P_{\text{source},j} - \kappa\Omega^2)/(I\Omega)$, $\alpha := 2\kappa/I$ and $K_{ij} := P_{\max,ij}/(I\Omega)$, this finally leads to the equation of motion

$$\frac{\mathrm{d}^2\phi_j}{\mathrm{d}t^2} = P_j - \alpha\frac{\mathrm{d}\phi_j}{\mathrm{d}t} + \sum_i K_{ij}\sin(\phi_i - \phi_j). \tag{A.5}$$

Unless stated otherwise, we assume that all transmission lines are equal, that is,

$$K_{i,j} = \begin{cases} K & \text{if a link exists between nodes } i \text{ and } j, \\ 0 & \text{otherwise.} \end{cases} \tag{A.6}$$

Appendix B. The analytic solution of the elementary model

Here we provide details of the derivation for the critical coupling strength K_c for the elementary network as given in equation (8). The starting point is the condition

$$0 = P_j + (K_{j+1}S_{j+1} - K_jS_j) \tag{B.1}$$

for the phase-locked steady state, see equation (5). Here, we have defined

$$S_j = \sin(\phi_j - \phi_{j-1}), \qquad j \in \{1, \ldots, 6\}, \tag{B.2}$$

identifying $\phi_0 = \phi_6$. Since one of the linear equations is redundant, the solutions span a one-dimensional space parametrized by a real number δ,

$$\vec{S} = \frac{P}{K}(\vec{S}_a - \delta\,\vec{S}_b) \tag{B.3}$$

with $\vec{S}_a = (1, 1, 0, -K/\tilde{K}, -1, 0)$ and $\vec{S}_b = (2, 1, 1, 2K/\tilde{K}, 1, 1)$.

The synchronized state is lost when the link between oscillators 4 and 5 becomes overloaded. The critical state is thus characterized by

$$S_5 = -1 \quad \text{for } K = K_c. \tag{B.4}$$

Using equation (B.3), we thus find that

$$S_5 = \frac{P}{K_c}(-1-\delta) = 1 \Rightarrow \delta = -1 + \frac{K_c}{P}. \tag{B.5}$$

Similarly, we write $\tilde{K} = (1+\epsilon)P$ with a small parameter ϵ. To leading order in δ and ϵ, we then find that

$$\begin{aligned} &S_1 = 1 - 3\delta + \cdots, \qquad S_2 = 1 - 2\delta + \cdots, \\ &S_{3,6} = -\delta + \cdots, \qquad S_4 = -2 - 2\delta + \epsilon + \cdots, \\ &S_5 = -1. \end{aligned} \tag{B.6}$$

Now we link the S_j to the phase differences $\Delta\phi_j = \phi_j - \phi_{j-1}$. Again we assume that the difference from the unperturbed network ($\epsilon = 0$) is small, i.e. we write

$$\Delta\phi_j = \Delta\phi_j(\epsilon = 0) + \alpha_j, \tag{B.7}$$

where α_j is small. We then find that

$$\begin{aligned} &\Delta\phi_{1,2}(\epsilon = 0) = \pi/2 \Rightarrow S_{1,2} = 1 - \frac{\alpha_{1,2}^2}{2} + \mathcal{O}(\alpha^4), \\ &\Delta\phi_{4,5}(\epsilon = 0) = -\pi/2 \Rightarrow S_{4,5} = -1 + \frac{\alpha_{4,5}^2}{2} + \mathcal{O}(\alpha^4), \\ &\Delta\phi_{3,6}(\epsilon = 0) = 0 \Rightarrow S_{3,6} = \alpha_{3,6} + \mathcal{O}(\alpha^3). \end{aligned} \tag{B.8}$$

We can now equate the S_j from equation (B.6) and from equation (B.8) to obtain the phase differences $\Delta\phi_j$ in terms of ϵ and δ. Now the phases differences must satisfy the cyclic path condition

$$\sum_j (\phi_j - \phi_{j-1}) = \sum_j \arcsin(S_j) = 0 \,(\mathrm{mod}\, 2\pi), \tag{B.9}$$

such that we obtain

$$\begin{aligned} &-\sqrt{6}\sqrt{\delta} - 2\sqrt{\delta} - 2\delta + \sqrt{-4\delta + 2\epsilon} = 0 \\ &\Rightarrow \epsilon = 2\delta + \tfrac{1}{2}[2\delta + (2+\sqrt{6})\sqrt{\delta}]^2. \end{aligned} \tag{B.10}$$

Appendix C. Numerical analysis of Braess's paradox in complex networks

In this section, we provide a detailed description of the numerical procedure used for analyzing the occurrence of Braess's paradox in complex networks. For every network, we calculate the order parameter r as a function of the coupling strength K. Starting from the strongly coupled case ($K = 100$), we calculate a steady state by solving the algebraic equation

$$P_j + \sum_i K_{ij} \sin(\phi_i - \phi_j) = 0 \tag{C.1}$$

using the MATLAB routine fsolve. For $K = 100$, the solver converges to the synchronous steady state where all phases are close to 0. We then lower the coupling strength adiabatically in small

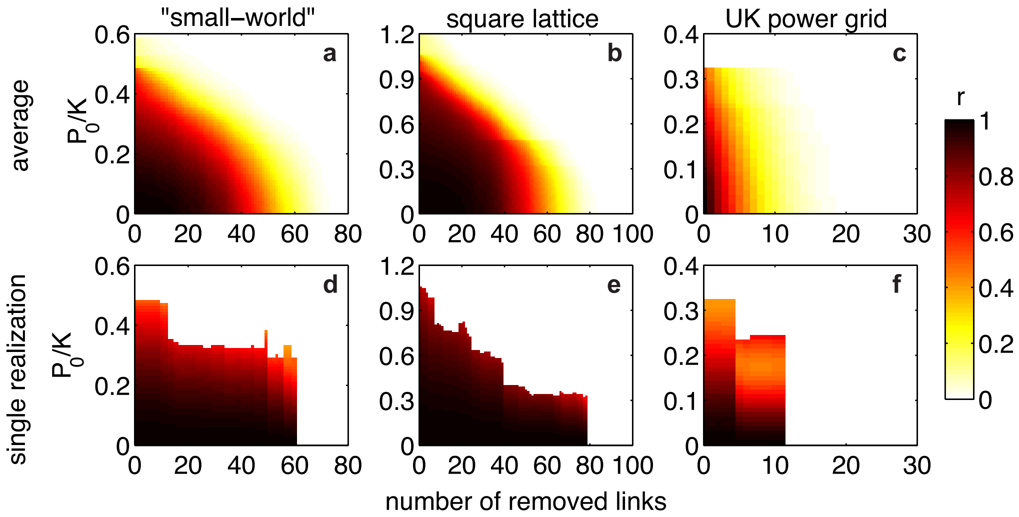

Figure C.1. Inverse percolation subject to Braess's paradox. Shown is the order parameter r as a function of $1/K$ and the number of removed links (a)–(c) averaged over 100 realizations and (d)–(f) for a single realization. We consider three different network models: (a), (d) a randomized ring network with $N = 100$ nodes, 200 links and topological randomness $q = 0.1$; (b), (e) a randomized square lattice with $N = 100$ nodes and $q = 0.1$; and (c), (f) the UK power grid model described in figure 3. For the small-world network and the square lattice, 50 nodes are randomly selected to be generators ($P_j = +P_0$) and 50 to be consumers ($P_j = -P_0$).

steps $\Delta(1/K) = 0.01 \times P_0^{-1}$ and calculate the steady state using the previous result as an initial guess. When a solution is found we calculate the steady-state order parameter

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{\mathrm{i}\phi_j}. \tag{C.2}$$

If no solution can be found we set $r = 0$ and stop. The critical coupling K_c is defined as the smallest value of K for which a steady state exists.

To systematically study how Braess's paradox depends on the topology of the power grid, we consider networks that interpolate between regular and random topology, also referred to as small-world networks [28, 29]. We start with either a square lattice with periodic boundary conditions or a ring with nearest and next-to-nearest neighbor coupling. Then every link is rewired with probability q, i.e. removed and re-inserted at a different randomly chosen position. In each case, we place the generators at randomly chosen nodes of the network. We consider homogeneously powered networks, where half of the nodes are generators ($P_j = +P_0$), and heterogeneously powered networks, where 20% of the nodes are generators ($P_j = 4\,P_0$). The remaining nodes are consumers with $P_j = -P_0$.

To identify Braess's paradox, we then *remove single links*. If this leads to a decrease of the critical coupling strength K_c, then the removal of the link promotes synchrony and the link is said to be subject to Braess's paradox. First we compute the fraction of links that show Braess's paradox. To this end we calculate the critical coupling K_c for a network where a single link is removed and compare it to K_c of the full network. The numerical results are shown in figure 5

Table C.1. Most networks exhibit Braess's paradox in the inverse percolation process. Results are given for a randomized square lattice and a randomized ring network [28] with $q = 0.1$, $N = 100$ nodes and 200 edges each.

	Showing Braess's paradox	
Network type	Fraction of networks	Fraction of links (%)
Randomized square lattice, homogeneous	981/1000	8.52
Randomized ring, homogeneous	928/1000	5.99
Randomized square lattice, heterogeneous	902/1000	4.77
Randomized ring, heterogeneous	753/1000	3.71

for the case of a randomized square lattice. As described above, the coupling strength is varied in steps of $\Delta(1/K) = 0.01 \times P_0$. Thus smaller changes in K_c are not detected.

In the inverse percolation problem (cf figures 6 and C.1), we remove links one by one. Starting from the full network we remove a link, calculate $r(K)$, remove another link, calculate $r(K)$ and so on until the network is no longer globally connected. Table C.1 summarizes the fraction of links that are subject to Braess's paradox during the inverse percolation process and the fraction of networks that show Braess's paradox at least once during the process. We observe that Braess's paradox is most common for a square lattice, as there are many small cycles that can suffer from geometric frustration. Furthermore, we find that Braess's paradox occurs consistently, but slightly less frequently, in heterogeneously powered networks.

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