On the Achievable Rate of Bandlimited Continuous-Time 1-Bit Quantized AWGN Channels

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***Abstract*—We consider a continuous-time bandlimited additive white Gaussian noise channel with 1-bit output quantization. On such a channel the information is carried by the temporal distances of the zero-crossings of the transmit signal. The set of input signals is constrained by the bandwidth of the channel and an average power constraint. Under a set of assumptions, we derive a lower bound on the capacity by lower-bounding the achievable rate for a given set of waveforms with exponentially distributed zero-crossing distances. We focus on the behaviour in the high signal-to-noise ratio regime and characterize the achievable rate depending on the available bandwidth and the signal-to-noise ratio.**

I. Introduction

For very high data rate short link communication the power consumption of the analog-to-digital converter (ADC) becomes a major factor, also compared to the transmit power. This is due to the required high quantization resolution and the very high sampling rate. One option to circumvent this is coarse quantization and oversampling at the receiver w.r.t. to the Nyquist rate, as one-bit quantization does not require highly linear analog signal processing. Obviously, optimal communication over the resulting channel requires an adapted modulation and signaling scheme as the information is carried in the zero-crossing time instants of the transmitted signal. The question is, how this affects the channel capacity compared to an additive white Gaussian noise (AWGN) channel quantized with high resolution and sampled at Nyquist rate. Previously, in [1] simulative approaches on bounding the achievable rate in a discrete-time scenario are studied. In [2], [3], the achievable rate is evaluated via simulation for different signaling strategies. For the noise free case, analytical approaches can be found already in [4] and [5], where it has been shown that oversampling can increase the information rate. Moreover, for the low signal-to-noise ratio (SNR) domain in [6] it was shown that oversampling increases the capacity per unit-cost of bandlimited Gaussian channels with 1-bit output quantization. In [7] it was proven that oversampling increases the achievable rate based on the study of the generalized mutual information.

An analytical evaluation of the channel capacity of the 1-bit quantized oversampled AWGN channel is still open. As a limiting case, in the present work, we study the capacity of the underlying continuous-time 1-bit quantized channel. Without time quantization, there is no quantization in the information carrying dimension. Its capacity is obviously upper-bounded by the AWGN channel as given by Shannon [8]. The channel in question corresponds to some extent to a timing channel as, e.g., studied in [9]. Given the outlined application scenario of short range multigigabit/s-communication, we focus on the mid to high SNR domain. We derive a lower bound on the capacity of the bandlimited continuous-time additive Gaussian noise channel with 1-bit output quantization. The derivation is based on certain approximations and simplifications, which will be clearly stated and are suitable in the mid to high SNR domain. We show that the achievable rate increases with the bandwidth for an appropriately chosen input distribution but saturates over the SNR. Moreover, we observe that the ratio between our lower bound and the AWGN capacity is a constant independent of the bandwidth for a given SNR and the appropriately chosen input distribution.

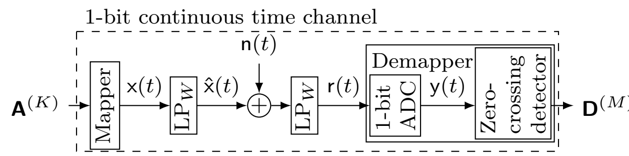

Fig. 1. System model

II. System Model and Design Parameters

We consider the system model depicted in Fig. 1. All information that can be conveyed through such a channel is encoded in the time instants of the zero-crossings.[1] Hence, the channel input and output vectors, $\mathbf{A}^{(K)} = [\mathsf{A}_1, ..., \mathsf{A}_K]^T$ and $\mathbf{D}^{(M)} = [\mathsf{D}_1, ..., \mathsf{D}_M]^T$ contain the temporal distances A_k and D_m of two consecutive zero-crossings of the transmit signal $\mathsf{x}(t)$ and the received signal $\mathsf{r}(t)$, respectively. Here, K is not necessarily equal to M as noise can add or remove zero-crossings. The corresponding processes are denoted $\mathbf{A}$ and $\mathbf{D}$.

We assume that the time instants of the zero crossings can be resolved with infinite precision, which makes A_k and D_m continuous random variables. We consider the input symbols A_k to be i.i.d. exponentially distributed with

$$\mathsf{A}_k \sim \lambda e^{-\lambda(a-\beta)} \mathbb{1}_{[\beta,\infty)}(a) \tag{1}$$

since this is entropy maximizing for continuous random variables supported on the interval $[\beta, \infty)$ with given mean. Here, $\mathbb{1}_{[u,v]}(x)$ is the indicator function. This results in a mean symbol duration of

$$T_{\text{avg}} = {}^1\!/\!_\lambda + \beta. \tag{2}$$

[1] Note that one additional bit is carried by the sign of the signal. However, its effect on the mutual information between channel input and output can be neglected when studying the capacity as it converges to zero for infinite blocklength.

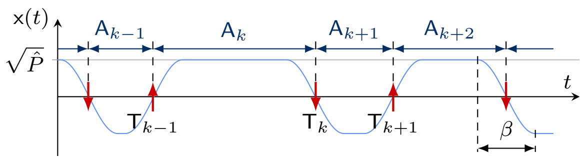

Fig. 2. Mapping from input sequence $\mathsf{A}^{(K)}$ to $\mathsf{x}(t)$

The mapper converts the random vector $\mathsf{A}^{(K)}$ into the continuous-time transmit signal $\mathsf{x}(t)$ as illustrated in Fig. 2. The signal $\mathsf{x}(t)$ alternates between $\pm\sqrt{\hat{P}}$ with zero-crossings at the times T_k and peak power $\hat{P}$. The transition between the levels is modeled by a cosine waveform, yielding

$$\mathsf{x}(t) = \left(\sum_{k=1}^{K} \sqrt{\hat{P}}(-1)^k g(t - \mathsf{T}_k)\right) + \sqrt{\hat{P}} \tag{3}$$

with the pulse shape

$$g(t) = \left(1 - \cos\left(\frac{\pi t}{\beta}\right)\right) \cdot \mathbb{1}_{[0,\beta]} + 2 \cdot \mathbb{1}_{[\beta,\infty)}. \tag{4}$$

The transition time β is chosen according to the available bandwidth W of the channel with

$$\beta = \frac{1}{2W}. \tag{5}$$

For $\lambda \to \infty$ this leads to a one sided signal bandwidth of W. However, $\mathsf{x}(t)$ is only almost bandlimited, with a small portion of its energy outside of the interval $[-W, W]$. Strict bandlimitation is ensured by the lowpass (LP) filters at transmitter and receiver, which are considered to be an ideal LP with one-sided bandwidth W and amplitude one. The LP-filtered signal $\hat{\mathsf{x}}(t)$ is transmitted over a continuous-time AWGN channel. The received signal after quantization and LP-filtering is given by $\mathsf{y}(t) = Q(\mathsf{x}(t) + \mathsf{z}(t))$ where $\mathsf{z}(t) = \hat{\mathsf{n}}(t) + \tilde{\mathsf{x}}(t)$ is the distortion introduced by the filtered white Gaussian noise $\hat{\mathsf{n}}(t)$ and the distortion $\tilde{\mathsf{x}}(t) = \mathsf{x}(t) - \hat{\mathsf{x}}(t)$ due to LP-filtering. Throughout our analysis, $\tilde{\mathsf{x}}(t)$ is approximated to be Gaussian. This enables closed form analytical treatment of the problem. Furthermore, $Q(\cdot)$ denotes a binary quantizer with threshold zero, i.e., $Q(x) = 1$ if $x \geq 0$ and $Q(x) = -1$ if $x < 0$.

The noise $\mathsf{n}(t)$ is zero-mean additive white Gaussian noise with power spectral density (PSD) $N_0/2$. Its filtered version is $\hat{\mathsf{n}}(\mathsf{t})$ has the PSD

$$S_{\hat{\mathsf{n}}}(f) = \begin{cases} N_0/2 & \text{for } |f| \leq W \\ 0 & \text{otherwise} \end{cases}. \tag{6}$$

Accordingly, the variance of the Gaussian noise is $\sigma_{\hat{\mathsf{n}}}^2 = N_0 W$ and the variance of $\tilde{\mathsf{x}}(t)$ is given by $\sigma_{\tilde{\mathsf{x}}}^2 = \mathbb{E}\left[|x(t) - \hat{x}(t)|^2\right]$. The signal-to-noise ratio is defined as

$$\rho = \frac{P}{N_0 W} \tag{7}$$

where P is the average power of $\mathsf{x}(t)^2$. It is given by

$$P = \frac{\hat{P}}{T_{\text{avg}}}\left(\int_0^{\beta} \cos^2\left(\frac{\pi}{\beta}t\right) dt + \frac{1}{\lambda}\right) = \frac{\frac{1}{2} + 2W\lambda^{-1}}{1 + 2W\lambda^{-1}}\hat{P}. \tag{8}$$

[2]Note, that the actual SNR is $\rho^* = (P - \sigma_{\tilde{\mathsf{x}}}^2)/(N_0 W) < \rho$. Hence, the results we obtain for ρ are actually achievable with ρ^*, s.t. the lower bound on the achievable rate is maintained. As we only obtain an upper bound on $\sigma_{\tilde{\mathsf{x}}}^2$, cf. Section VI, we do not know ρ^* and, thus, define the SNR as ρ in (7).

III. Achievable Rate and Error Events

The capacity of the communication channel is defined as the supremum of the mutual information rate over all input distributions with constrained average power P and bandwidth W. The mutual information rate is hereby defined as

$$I'(\mathbf{A}; \mathbf{D}) = \lim_{K \to \infty} \frac{1}{K T_{\text{avg}}} I\left(\mathbf{A}^{(K)}; \mathbf{D}^{(M)}\right) \tag{9}$$

with $I\left(\mathbf{A}^{(K)}; \mathbf{D}^{(M)}\right)$ being the mutual information. We omit the evaluation of the supremum, but derive a lower bound on the capacity based on the input signal described in Section II.

The channel can alter the zero-crossings of the received signal $\mathsf{r}(t)$ w.r.t. $\mathsf{x}(t)$ by shifts, leading to an error in magnitude of A_k, or a pair of zero-crossings can be either introduced or deleted, leading to insertion or deletion of symbols. Insertion and deletion channels have been studied to the best of our knowledge only for binary channels via combinatorial approaches, e.g., [10], [11]. For the considered input signals and the high SNR scenario, we neglect deletions since $\mathsf{A}_k \geq \beta$. In this regard consider, that two sampling points of the band-limited noise with distance β can be considered uncorrelated and, thus, the probability of the noise remaining high over the complete duration of $A_k \geq \beta$ is low, cf. [12, Appendix E].

The remaining error events, are shifts denoted below by the process $\mathbf{S}$ and insertions denoted by the process $\mathbf{V}$ of zero-crossings. Both can be analyzed separately using the idea of a genie-aided receiver as in [11]. We provide the information $\mathbf{V}$ to the receiver, such that it can remove the additional zero-crossings. The resulting channel output is $\hat{\mathbf{D}}$. For the mutual information rate in case the receiver has the side information about the inserted zero-crossings it holds

$$I'(\mathbf{A}; \hat{\mathbf{D}}) = I'(\mathbf{A}; \mathbf{D}, \mathbf{V}). \tag{10}$$

By applying the chain rule to (10), we can write for the mutual information rate in (9)

$$I'(\mathbf{A}; \mathbf{D}) = I'(\mathbf{A}; \mathbf{D}, \mathbf{V}) - I'(\mathbf{A}; \mathbf{V}|\mathbf{D}). \tag{11}$$

For the characterization of the auxiliary process $\mathbf{V}$, we consider the transmission of one input symbol A_k. Its bounding zero-crossings T_{k-1} and T_k will be shifted to $\hat{\mathsf{T}}_{k-1}$ and $\hat{\mathsf{T}}_k$ by the noise process such that

$$\hat{\mathsf{T}}_k = \mathsf{T}_k + \mathsf{S}_k \tag{12}$$

where S_k is a shift in time caused by the noise $\mathsf{z}(t)$. Furthermore, additionally introduced zero-crossings will divide the input symbol into a vector of corresponding received symbols. The latter is reversible, if the receiver knows which zero-crossings correspond to the originally transmitted symbol. The receiver could sum up all those received distances D_m in order to obtain $\hat{\mathsf{D}}_k$. Hence, starting from the first received symbol onwards, the auxiliary sequence $\mathbf{V}^{(K)}$ consists of positive integer numbers $\mathsf{V}_k \in \mathbb{N}$, representing for each input symbol the number of corresponding output symbols. As $\mathbf{V}$ is discrete, we can bound the information rate in (11) by

$$\begin{aligned} I'(\mathbf{A}; \mathbf{D}) &= I'(\mathbf{A}; \mathbf{D}, \mathbf{V}) - H'(\mathbf{V}|\mathbf{D}) + H'(\mathbf{V}|\mathbf{D}, \mathbf{A}) \\ &\geq I'(\mathbf{A}; \mathbf{D}, \mathbf{V}) - H'(\mathbf{V}) \end{aligned} \tag{13}$$

where (13) results from the fact that the entropy rate is non-negative and that conditioning cannot increase entropy.

The proof of the existence of a coding theorem remains for future research. In the following, we will derive bounds on $I'(\mathsf{A};\mathsf{D},\mathsf{V})$ and $H'(\mathsf{V})$.

IV. Achievable Rate of the Genie-Aided Receiver

To evaluate $I'(\mathsf{A};\hat{\mathsf{D}})$ of the genie-aided receiver, cf. (13) and (10), we have to evaluate the mutual information rate between the sequence of temporal spacings between the zero-crossings of the input signal $\mathsf{A}^{(K)}$ and genie-aided channel output $\hat{\mathsf{D}}^{(K)}$. Note, that in contrast to the original channel, here both vectors are of same length. The only error remaining is a shift S_k of every zero-crossings instant T_k to $\hat{\mathsf{T}}_k$. On a symbols level we can write for the channel output

$$\hat{\mathsf{D}}_k = \hat{\mathsf{T}}_k - \hat{\mathsf{T}}_{k-1} = \mathsf{A}_k + \mathsf{S}_k - \mathsf{S}_{k-1} \tag{14}$$

For the given system model, two assumptions are reasonable

(A1) the shifts S_k are independent

(A2) there is only one zero-crossing in each transition interval $\left[\mathsf{T}_k - \frac{\beta}{2}, \mathsf{T}_k + \frac{\beta}{2}\right]$

Assumption (A1) is due to the fact that any S_k and S_{k-1} are spaced at least time β apart, which is above the coherence time of the noise. Likewise, due to the bandlimitation of the noise additional zero-crossings are very unlikely within the transition interval leading to (A2). Further details are given in [12].

A. The Distribution of the Shifting Errors

The distribution of the S_k can be evaluated by mapping the probability density function of the additive noise $\mathsf{z}(\mathsf{T}_k)$ at the time instant T_k by the function

$$\mathsf{z}(\mathsf{T}_k) = -\sqrt{\hat{P}}\sin\left(\frac{\pi}{\beta}\mathsf{S}_k\right) \tag{15}$$

into the zero-crossing error S_k on the time axis. The mapping depends on the slope of the transition waveform. As $\mathsf{r}(t)$ is bandlimited, it can be described adequately by a sampled representation with sampling rate $1/\beta$ to fulfill the Nyquist condition, cf. (5). Note that we refer to sampling only to evaluate the value of $\mathsf{z}(t)$ at the certain time instant T_k of the original zero-crossing. We still assume the receiver to be able to resolve the zero-crossing time instants with infinite resolution. As the exact distributions of $\tilde{\mathsf{x}}(t)$ is unknown, we approximate it by a Gaussian distribution with the same mean and variance. As $\hat{\mathsf{n}}(t)$ and $\tilde{\mathsf{x}}(t)$ are independent, we obtain

$$\sigma_{\mathsf{z}}^2 = \sigma_{\hat{\mathsf{n}}}^2 + \sigma_{\tilde{\mathsf{x}}}^2 = N_0 W + \sigma_{\tilde{\mathsf{x}}}^2 \tag{16}$$

and $\mathsf{z}(t) \sim \mathcal{N}(0, \sigma_{\mathsf{z}}^2)$. As we are focusing on the mid to high SNR behaviour of the capacity, the zero-crossing errors S_k are with high probability small in comparison to the transition time β such that we can assume $s/\beta \ll 1$. Hence, the zero-crossing errors S_k can be approximated to be zero-mean Gaussian distributed with variance

$$\sigma_{\mathsf{S}}^2 = \frac{\sigma_{\mathsf{z}}^2}{4\pi^2 W^2 \hat{P}}. \tag{17}$$

The validity of the high SNR assumption is analyzed in [12, Appendix B]. It holds for SNR of approximately $\rho \geq 6\,\text{dB}$.

B. Lower Bound on $I'(\mathsf{A};\hat{\mathsf{D}})$

The mutual information between the temporal spacings of the zero-crossings of the channel input signal $\mathsf{A}^{(K)}$, and the zero-crossings of the signal at the output of the genie-aided receiver $\hat{\mathsf{D}}^{(K)}$ is given by

$$\begin{aligned} I\left(\mathsf{A}^{(K)};\hat{\mathsf{D}}^{(K)}\right) &= h\left(\mathsf{A}^{(K)}\right) - h\left(\mathsf{A}^{(K)}|\hat{\mathsf{D}}^{(K)}\right) \\ &= h\left(\mathsf{A}^{(K)}\right) - h\left(\mathsf{A}^{(K)} - \hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}\middle|\hat{\mathsf{D}}^{(K)}\right) \end{aligned} \tag{18}$$

where $h(\cdot)$ denotes the differential entropy. Moreover, $\hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}$ is the linear minimum mean-squared error estimate of $\mathsf{A}^{(K)}$ based on $\hat{\mathsf{D}}^{(K)}$. Equality (18) follows from the fact that addition of a constant does not change differential entropy and the fact that $\hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}$ can be treated as a constant while conditioning on $\hat{\mathsf{D}}^{(K)}$. Next, we will upper-bound the second term on the RHS of (18). This term describes the randomness of the linear minimum mean-squared estimation error while estimating $\mathsf{A}^{(K)}$ based on the observation $\hat{\mathsf{D}}^{(K)}$. It can be upper-bounded by the differential entropy of a Gaussian random variable having the same covariance matrix [13, Th. 8.6.5]. The estimation error covariance matrix of the linear minimum mean-squared error (LMMSE) estimator is given by

$$\begin{aligned} \mathbf{R}_{\text{err}}^{(K)} &= \mathbb{E}\left[\left(\mathsf{A}^{(K)} - \hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}\right)\left(\mathsf{A}^{(K)} - \hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}\right)^T\right] \\ &= \sigma_{\mathsf{A}}^2\mathbf{I}^{(K)} - \sigma_{\mathsf{A}}^4\left(\sigma_{\mathsf{A}}^2\mathbf{I}^{(K)} + \sigma_{\mathsf{S}}^2\mathbf{R}_{\mathbf{S}}^{(K)}\right)^{-1} \end{aligned} \tag{19}$$

where

$$\sigma_{\mathsf{A}}^2\mathbf{I}^{(K)} = \mathbb{E}\left[\left(\mathsf{A}^{(K)} - \boldsymbol{\mu}_{\mathsf{A}}\right)\left(\mathsf{A}^{(K)} - \boldsymbol{\mu}_{\mathsf{A}}\right)^T\right] = \lambda^{-2}\mathbf{I}^{(K)} \tag{20}$$

with $\boldsymbol{\mu}_{\mathsf{A}} = \mathbb{E}\left[\mathsf{A}^{(K)}\right] = \left(\beta + \lambda^{-1}\right)\mathbf{1}^{(K)}$, cf. (2), and where (20) follows from the fact that the elements of $\mathsf{A}^{(K)}$ are independent exponentially distributed, see (1). Furthermore, $\mathbf{I}^{(K)}$ is the identity matrix of size $K \times K$ and $\mathbf{1}^{(K)}$ is the all one column vector of length K. Moreover, $\sigma_{\mathsf{S}}^2\mathbf{R}_{\mathbf{S}}^{(K)}$ is the covariance matrix of the shifting error, cf. (14), i.e.,

$$\sigma_{\mathsf{S}}^2\mathbf{R}_{\mathbf{S}}^{(K)} = \sigma_{\mathsf{S}}^2 \begin{pmatrix} 2 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \ddots & \vdots \\ 0 & -1 & 2 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -1 \\ 0 & \dots & 0 & -1 & 2 \end{pmatrix} \tag{21}$$

of size $K \times K$. Here, we have used that $\mathsf{S}_k - \mathsf{S}_{k-1}$ is zero-mean. Thus, we get

$$h(\mathsf{A}^{(K)} - \hat{\mathsf{A}}_{\text{LMMSE}}^{(K)}|\hat{\mathsf{D}}^{(K)}) \leq \frac{1}{2}\log\det\left(2\pi e\mathbf{R}_{\text{err}}^{(K)}\right) \tag{22}$$

yielding for the mutual information in (18)

$$\begin{aligned} I(\mathsf{A}^{(K)};\hat{\mathsf{D}}^{(K)}) &\geq K h(\mathsf{A}_k) \\ &+ \frac{1}{2}\log\det\left((2\pi e)^{-1}\left(\sigma_{\mathsf{A}}^{-2}\mathbf{I}^{(K)} + \sigma_{\mathsf{S}}^{-2}(\mathbf{R}_{\mathsf{S}}^{(K)})^{-1}\right)\right) \end{aligned} \tag{23}$$

where the first term of (23) follows as the A_k are i.i.d. and for the second term we have used (19) and the matrix inversion lemma. With (23) the mutual information rate in (10) is lower-bounded by

$$I'(\mathbf{A};\hat{\mathbf{D}}) \geq \frac{h(\mathsf{A}_k)}{T_{\text{avg}}} + \frac{1}{2T_{\text{avg}}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \log \frac{\sigma_{\mathsf{A}}^{-2} + \sigma_{\mathsf{S}}^{-2} S_{\mathsf{S}}^{-1}(f)}{2\pi e} df \quad (24)$$

where for (24) we have used Szegö's theorem on the asymptotic eigenvalue distribution of Hermitian Toeplitz matrices [14, pp. 64-65], [15] with $S_{\mathsf{S}}(f)$ being the PSD corresponding to the sequence of covariance matrices $\mathbf{R}_{\mathsf{S}}^{(K)}$. It is given by

$$S_{\mathsf{S}}(f) = 2(1-\cos(2\pi f)). \qquad |f| < 0.5. \quad (25)$$

As the A_k are exponentially distributed, we get $h(\mathsf{A}_k) = 1 - \log(\lambda)$. With this and (2), (5), (17), (20), the lower bound in (24) can be written as

$$I'(\mathbf{A};\hat{\mathbf{D}}) \geq \frac{1}{2T_{\text{avg}}} \left\{ \log\left(\frac{e}{2\pi}\right) + \operatorname{arcosh}\left(\frac{1}{2\sigma_{\mathsf{S}}^2\lambda^2} + 1\right) \right\}$$
$$= \frac{W}{1+2W\lambda^{-1}} \left\{ \log\left(\frac{e}{2\pi}\right) + \operatorname{arcosh}\left(\frac{2\pi^2 W^2 \hat{P}}{\sigma_{\mathsf{z}}^2 \lambda^2} + 1\right) \right\}. \quad (26)$$

V. The Process of Additional Zero-Crossings

In order to bound the rate $I'(\mathbf{A};\mathbf{D})$ without the side information $\mathbf{V}$ provided to the receiver, it remains to find an explicit expression or an upper bound for $H'(\mathbf{V})$, cf. (13). For every input symbol A_k the random variable V_k, describing the number of received symbols corresponding to a transmit symbol, depends on the number N_k of introduced zero-crossings by $\mathsf{V}_k = \mathsf{N}_k + 1$. Based on (A2) we do not need to consider the transition intervals as they just contain the shifted zero-crossings. It remains the time $T_{\text{sat}} = \mathbb{E}[\mathsf{A}_k] - \beta = \lambda^{-1}$ in which the signal level $\pm\sqrt{\hat{P}}$ is maintained, leading to a level-crossing problem, which has been widely studied, e.g., in [16], [17]. Leveraging those results, we derive a bound on $H'(\mathbf{V})$ based on $\mathbb{E}[\mathsf{V}_k]$. By applying the Rice formula [16], we obtain

$$\mu = \mathbb{E}[\mathsf{V}_k] = \frac{1}{\pi}\sqrt{\frac{-s''_{\mathsf{zz}}(0)}{\sigma_{\mathsf{z}}^2}} \exp\left(-\frac{\hat{P}}{2\sigma_{\mathsf{z}}^2}\right)\lambda^{-1} + 1. \quad (27)$$

Here, $s_{\mathsf{zz}}(\tau)$ is the autocorrelation function (ACF) of $\mathsf{z}(t)$, which is approximated as a Gaussian process as stated above, and $s''_{\mathsf{zz}}(\tau) = \frac{\partial}{\partial\tau^2} s_{\mathsf{zz}}(\tau)$. For $-s''_{\mathsf{zz}}(0) < \infty$, it holds that $\mu < \infty$. Analogously to (16), we get

$$s''_{\mathsf{zz}}(0) = -\frac{4}{3}N_0 W^3 + s''_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(0) \quad (28)$$

where $s''_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(0)$ is finite for finite bandwidths W, cf. Section VI. We upper-bound $H'(\mathbf{V})$ based on the entropy of a geometric distribution with mean μ, cf. (27) and [12, Appendix C], using

$$H(\mathsf{V}_k) \leq (1-\mu)\log(\mu-1) + \mu\log\mu. \quad (29)$$

All time intervals with maximum signal level $\pm\sqrt{\hat{P}}$ are spaced by the transition time β apart and, hence, the V_k can be considered i.i.d. Thus, for the entropy rate $\mathbf{V}$ of we get

$$H'(\mathbf{V}) = \frac{1}{T_{\text{avg}}} H(\mathsf{V}_k). \quad (30)$$

Note that the bound on $H(\mathsf{V}_k)$ is an increasing function in μ. Furthermore, the expected number of level-crossings of a random Gaussian process increases with its variance. Hence, to upper-bound (27) an upper bound for σ_{z}^2 and, thus, for $\sigma_{\tilde{\mathsf{x}}}^2$ is required. An upper bound on $\sigma_{\tilde{\mathsf{x}}}^2$ results in a lower bound on $s''_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(0)$, cf. Section VI, as the two parameters depend on the ACF of the noise process and cannot be chosen independently. Both bounds will be derived in the next section.

VI. Signal Distortion by Lowpass-Filtering

The distortion $\tilde{\mathsf{x}}(t)$ of $\mathsf{x}(t)$ introduced by the LP-filter can be quantified by the clipped energy, using the mean squared error $\sigma_{\tilde{\mathsf{x}}}^2$ as distortion measure. As we consider a rectangular filter with cutoff-frequency W, the PSD of $\tilde{\mathsf{x}}(t)$ is $S_{\tilde{\mathsf{X}}}(f) = S_{\mathsf{X}}(f)$ for $|f| > W$ and zero otherwise. Applying Parseval's Theorem, we obtain

$$\sigma_{\tilde{\mathsf{x}}}^2 = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \mathbb{E}\left[\tilde{\mathsf{x}}^2(t)\right] dt \quad = \frac{1}{\pi}\int_{2\pi W}^{\infty} S_{\mathsf{X}}(\omega) d\omega \quad (31)$$

as $S_{\mathsf{X}}(\omega)$ is even. The PSD $S_{\mathsf{X}}(\omega)$ is given by [12, Section VI]

$$S_{\mathsf{X}}(\omega) = \lim_{K\to\infty} \frac{\mathbb{E}\left[|\mathsf{X}(\omega)|^2\right]}{KT_{\text{avg}}} \quad (32)$$
$$= \frac{2\hat{P}(1+\cos(\omega\beta))}{T_{\text{avg}}} \left[\frac{\pi^2}{\omega(\pi^2-\omega^2\beta^2)}\right]^2$$
$$\times \left(1 + 2\lim_{K\to\infty}\sum_{n=1}^{K-1}(-1)^n\left(1-\frac{n}{K}\right)\mathbb{E}[\cos(\omega \mathsf{L}_n)]\right) \quad (33)$$

where $n = k-j$ is the index describing the distance between two arbitrary zero-crossing instances and $\mathsf{L}_n = \mathsf{T}_k - \mathsf{T}_j$ is the corresponding random variable with probability distribution

$$p_{\mathsf{L}}(l_n) = \frac{\lambda^n e^{-\lambda(l_n - n\beta)}(l_n - n\beta)^{n-1}}{(n-1)!}, \; n \geq 1, l_n \geq n\beta. \quad (34)$$

The infinite sum in (33) is upper-bounded by [12, Section VI]

$$c(\omega) = \frac{\lambda}{\sqrt{\lambda^2+\omega^2} - \lambda}. \quad (35)$$

Hence, the PSD can be bounded as

$$S_{\mathsf{X}}(\omega) \leq \frac{2\hat{P}(1+2c(\omega))}{T_{\text{avg}}}(1+\cos(\omega\beta))\left[\frac{\pi^2}{\omega(\pi^2-\omega^2\beta^2)}\right]^2. \quad (36)$$

With (36) bounds on $\sigma_{\tilde{\mathsf{x}}}^2$ and $s''_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(0)$ can be computed. For $\sigma_{\tilde{\mathsf{x}}}^2$ we get with (31) and (36)

$$\sigma_{\tilde{\mathsf{x}}}^2 \leq \frac{(1+2c_1)\hat{P}\beta}{2T_{\text{avg}}\pi^2}\left[3\operatorname{Ci}(2\pi) - 3\gamma - 3\log(2\pi)\right.$$
$$\left. -\pi^2 + 4\pi\operatorname{Si}(\pi) - \pi\operatorname{Si}(2\pi)\right] \quad (37)$$

where $\gamma \approx 0.577$ is the Euler-Mascheroni constant, $\operatorname{Si}(\cdot)$ and $\operatorname{Ci}(\cdot)$ are the sine- and cosine-integral functions, and $c_1 = \lambda/(\sqrt{\lambda^2+4\pi^2W^2} - \lambda)$ involving a further bounding step for $|\omega| \geq W$ on the monotone decreasing function $c(\omega)$ in (35). Furthermore, the second derivative of $s_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(\tau)$ is given by

$$s''_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(\tau) = \frac{\partial^2}{\partial\tau^2} s_{\tilde{\mathsf{x}}\tilde{\mathsf{x}}}(\tau) = \frac{1}{\pi}\int_{2\pi W}^{\infty} S_{\mathsf{X}}(\omega)\frac{\partial^2}{\partial\tau^2}\cos(\omega\tau)d\omega \quad (38)$$

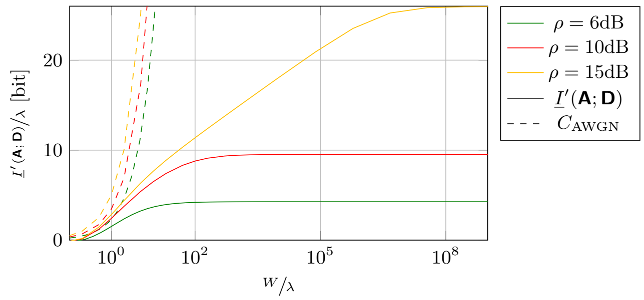

Fig. 3. Lower bound on $I'(\mathbf{A};\mathbf{D})$ in comparison to the AWGN capacity

where the exchangeability of differentiation and integration has been shown via Lebesgue's dominated convergence theorem [18, Theorem 1.34], with the dominating function $g(\omega) = \omega^2 S_{\mathsf{X}}(\omega)$. As in (38) $\frac{\partial^2}{\partial\tau^2}\cos(\omega\tau)\big|_{\tau=0} = -\omega^2$ and since $S_{\mathsf{X}}(\omega)$ is positive for all ω, an upper bound on $S_{\mathsf{X}}(\omega)$ results in a lower bound on $s''_{\mathsf{XX}}(0)$ given by

$$s''_{\mathsf{XX}}(0) \geq -\frac{(1+2c_1)\hat{P}}{2T_{\text{avg}}\beta}[\pi^2 - \gamma - \log(2\pi) - \pi\,\mathrm{Si}(2\pi) + \mathrm{Ci}(2\pi)]. \quad (39)$$

VII. Lower Bound on the Achievable Rate

Substituting (2), (5), (10), (26), (29), and (30) into (13), a lower bound on the achievable rate of the 1-bit quantized time continuous channel is given by

$$I'(\mathbf{A};\mathbf{D}) \geq \underline{I}'(\mathbf{A};\mathbf{D}) = \frac{2W}{2W\lambda^{-1}+1}\left[\frac{1}{2}\log\left(\frac{e}{2\pi}\right) + \frac{1}{2}\operatorname{arcosh}\left(\frac{2\pi^2W^2\hat{P}}{\sigma_{\mathsf{Z}}^2\lambda^2}+1\right) + \mu\log\left(\frac{\mu-1}{\mu}\right) - \log(\mu-1)\right] \quad (40)$$

with the equalities and inequalities (16), (27), (28), (37), and (39). Fig. 3 shows the lower bound in (40) for different SNRs ρ, see (7), where both axis are normalized by λ.

The achievable rate saturates for high bandwidths W due to the limited randomness of the input signal controlled by λ. The average symbol duration A_k is then large compared to the coherence time of the noise such that the expected number of additional zero-crossings within A_k becomes significant. For comparison also the capacity of the AWGN channel without output quantization is given, which is an upper bound to the capacity of the continuous-time 1-bit quantized channel studied in the present paper. It can be seen, that the lower bound is relatively tight for W/λ in the order of 1. In order to avoid saturation of the achievable rate by the chosen input distribution, the randomness of the input signal needs to be matched to the channel bandwidth, which is achieved by allowing λ to grow linearly with W, i.e., fixing the operation point on the abscissa in Fig. 3. It can be shown that $\underline{I}'(\mathbf{A};\mathbf{D})$ is a constant fraction of C_{AWGN} in case W/λ and ρ are fixed.

Furthermore, Fig. 4 shows that the lower bound $\underline{I}'(\mathbf{A};\mathbf{D})$ saturates for increasing SNR. Differently, the AWGN capacity increases logarithmically with the SNR without bound.

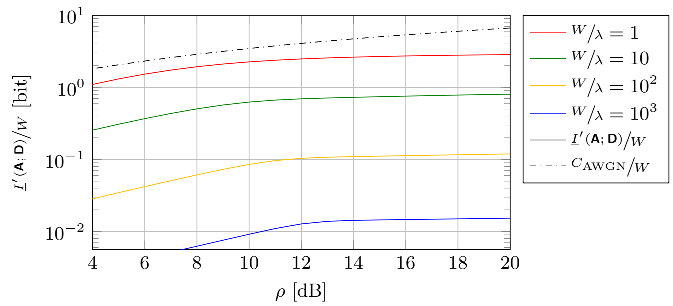

Fig. 4. Lower bound on $I'(\mathbf{A};\mathbf{D})$ depending on the SNR ρ

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