ON THE TIMING SYNCHRONIZATION UNDER 1-BIT QUANTIZATION AND OVERSAMPLING

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ABSTRACT

As the demand for communication systems with high data rates is increasing, large bandwidths, and thus high sampling rates, are required. As a consequence, the energy consumption of conventional high resolution analog-to-digital converters increases drastically. On the contrary, high resolution in time domain is less difficult to achieve than high resolution in amplitude domain. This motivates the design of communication systems with 1-bit quantization and oversampling. It has been shown that utilizing run-length limited sequences and faster-than-Nyquist signaling is beneficial in terms of achievable rate. However, it is an open question how receiver synchronization can be performed in such systems.

In this work we assume perfect frame, frequency and phase synchronization and investigate the effect of a fixed but unknown time shift. Due to 1-bit quantization, standard timing estimation and interpolation cannot be applied. We show that oversampling w.r.t. the signaling rate compensates for the error introduced by the time shift. If the oversampling factor is an integer value, estimating the time shift becomes obsolete if the oversampling rate is sufficiently high.

1. INTRODUCTION

The continued demand for faster communication systems requires data rates of multiple gigabit per second. Such high data rates imply high bandwidths and thus impose challenging requirements on the analog-to-digital converter (ADC). In particular in wireless short range scenarios, e.g., communication between computer boards [1,2] an ADC with multiple gigasamples per second has a major impact on the overall power consumption of the wireless link. Surveys show that power limited high sampling rates come at the price of coarse quantization [3]. Considering this, using an ADC with 1-bit quantization can be beneficial as the low resolution can be compensated by higher sampling rates. Since 1-bit quantization does neither need an automatic gain control, nor linear amplification, it is expected that this is still more energy efficient.

In [4], numerical studies have shown that sequence design and faster-than-Nyquist (FTN) signaling is beneficial in terms of achievable rate. Especially the utilization of run-length limited (RLL) sequences is an appropriate choice in terms of spectral efficiency. The results were extended to strictly bandlimited channels in [5]. A lower bound on the achievable rate of the continuous time (i.e., infinite oversampling) additive white Gaussian noise (AWGN) channel with 1-bit output quantization and strict bandlimitation was derived in [6].

Results on signal parameter estimation under 1-bit quantization can be found in [7–10]. The problem of channel state estimation with low precision quantization (1-3 bits) is investigated in [11]. Joint phase and frequency synchronization of a QPSK and Nyquist rate based communication system with coarse phase quantization was considered in [12]. The phase quantization can be implemented by passing linear combinations of the in-phase and quadrature components through 1-bit ADCs. Quantization into $2n$ phase bins requires n such linear combinations, and thus n 1-bit ADCs. Hence, the energy consumption is $\frac{n}{2}$ times higher compared to conventional 1-bit quantization with one 1-bit ADC in I and Q, respectively.

However, the design of timing synchronization algorithms with 1-bit quantization at the receiver is still open. Thus, in the present paper we will study the timing synchronization of a bandlimited communication system based on RLL sequences, FTN signaling and 1-bit quantization at the receiver. Here we assume perfect frame, phase and frequency synchronization. In conventional receivers with high resolution quantization, a timing error can be handled fully digitally via interpolation [13, Chapter 4]. We show that oversampling w.r.t. the signaling rate compensates for the errors introduced by a time shift. Thus, like in a conventional digital receiver, a voltage controlled oscillator (VCO) for sampling time adaptation is not required.

2. SYSTEM MODEL

The system model is depicted in Fig. 1. Since a receiver that relies on 1-bit quantization can only distinguish if the input signal is smaller or larger than zero, all information is conveyed in the temporal distances of the zero crossings of the signal. Therefore, we encode the information in run-length limited (RLL) sequences $\mathbf{x} = [x_1, x_2, \dots x_K]^T$. The elements of $\mathbf{x}$ are called transmit symbols. Furthermore, we define the vector $\mathbf{l} = [l_1, l_2, \dots, l_M]$ that consists of the run-lengths in the RLL sequence $\mathbf{x}$. Thus, the position of the mth level change in $\mathbf{x}$ is defined as $T_m = \sum_{i=1}^{m} l_i$. The digital to analog converter (DAC) maps the RLL sequences into an ideal rectangular analog signal

$$x(t) = \mathbb{1}(t) + 2\sum_{m=1}^{M}(-1)^m \mathbb{1}\left(t - \frac{\beta}{M_{\text{Tx}}}T_m\right), \qquad (1)$$

where $\mathbb{1}(t)$ is the Heaviside step function, β is the time of a Nyquist interval corresponding to the single sided channel bandwidth W and M_{Tx} is the FTN signaling factor, i.e., the number of symbols that are transmitted within one Nyquist

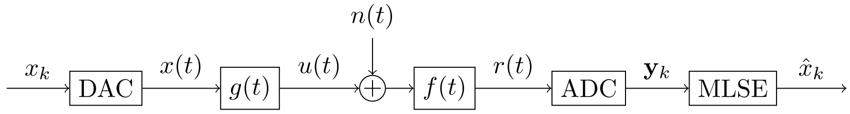

Fig. 1: System Model

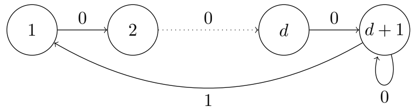

Fig. 2: State machine of a d contrained sequence

interval. The transmit filter $g(t)$ is a root raised cosine (RRC) filter with roll-off factor α and cut-off frequency $f_c = W = \frac{1}{2\beta}$, i.e., the channel bandwidth is defined in terms of the cut-off frequency and is independent of the roll-off factor. The transmit signal is defined as

$$u(t) = x(t)*g(t) = g_s(t)+2\sum_{m=1}^{M}(-1)^m g_s\left(t-\frac{\beta}{M_{\text{Tx}}}T_m\right), \tag{2}$$

where $g_s(t) = \int_{-\infty}^{t} g(\tau)d\tau$ is the step response of the transmit filter. The receive filter $f(t)$ is equal to the transmit filter. With $h(t) = g(t) * f(t)$ and $h_s(t) = \int_{-\infty}^{t} h(\tau)d\tau$ the output of the receive filter is given by

$$\begin{aligned} r(t) &= h_s(t) + 2\sum_{m=1}^{M}(-1)^m h_s(t-\frac{\beta}{M_{\text{Tx}}}T_m) + \eta(t) \\ &= s(t) + \eta(t) \end{aligned}, \tag{3}$$

where $\eta(t) = f(t) * n(t)$ with the white Gaussian noise $n(t)$. The samples of $r(t)$ are defined as

$$r_n = r\left(nT_s + \frac{T_s}{2} + \epsilon T_s\right), \epsilon \in [-0.5, 0.5], \tag{4}$$

where $T_s = \frac{\beta}{M_{\text{Tx}}M_{\text{Rx}}}$ is the time between two samples, ϵ is a fixed time shift and M_{Rx} is an additional oversampling factor w.r.t. the signaling rate that is required to cope with ϵ. Consider for now that M_{Rx} is an integer value. The output of the 1-bit ADC is given by

$$\begin{aligned} \mathbf{y}_k &= \text{sign}(\mathbf{r}_k) \\ &= \text{sign}\left([r_{kM_{\text{Rx}}+1}, r_{kM_{\text{Rx}}+2}, \ldots, r_{kM_{\text{Rx}}+M_{\text{Rx}}}]\right) \end{aligned}, \tag{5}$$

where $\text{sign}(\cdot)$ is the signum function. That implies that for every transmit symbol x_k there is a vector $\mathbf{y}_k$ of length M_{Rx}. Note that the noise samples $\eta(nT_s + \frac{T_s}{2} + \epsilon T_s)$ are correlated due to oversampling w.r.t. the Nyquist rate.

3. RUN-LENGTH LIMITED SEQUENCES

As the information is conveyed in the temporal distances of the zero-crossings, RLL sequences are a natural choice for modulation. RLL sequences are known from recording systems and some of the main results are summarized in [14]. An RLL sequence can be obtained from a (d, k) sequence where a one is followed by at least d and at most k zeros. The k constraint was introduced for practical reasons in recording systems, such as clock recovery. In this work we neglect the k constraint. Fig. 2 depicts the state machine of a d sequence. A d sequence can be transferred into an RLL sequence by non-return-to-zero-inverse (NRZI) encoding. This encoding produces a sequence with a sign flip whenever a one occurs in the d sequence, i.e., the $d = 1$ sequence

$$010001010000001001 \tag{6}$$

would be converted to the RLL sequence

$$1-1-1-1-111-1-1-1-1-1-1-1111-1 \tag{7}$$

It can easily be verified that an RLL sequence derived from a d sequence has at least $d+1$ consecutive identical symbols.

According to [15], the maximum entropy rate of such a sequence, also called code capacity, is given by

$$C(d) = \log_2\lambda, \tag{8}$$

where λ is the largest eigenvalue of the adjacency matrix of the state machine of the d sequence. Values are given in Table 1.

In practice, a simple method to obtain RLL sequences is by utilizing a fixed length block code that maps m information bits onto n code bits, i.e., the rate of the code is $R = m/n$. To obtain the numerical results presented in this work, we utilized a block code with the parameters $d = 2$, $m = 7$, $n = 14$ and hence $R = 1/2$. This results in a code efficiency $R/C(d) \approx 0.91$. The code was designed such that the codewords can be concatenated without violating the $d = 2$ constraint, by giving every codeword two leading zeros. The remaining $n - 2 = 12$ bits are exactly the 128 codewords that meet the $d = 2$ constraint [14], except the all zero codeword.

4. INTERFERENCE AND SPECTRAL EFFICIENCY

Depending on the roll-off factor, the transmit pulse $g_s(t)$ requires at least a time of β from a negative peak to a positive peak, or vice versa. Thus, to limit the interference two zero crossings should be at least β apart from each other. Since there are M_{Tx} transmit symbols within a Nyquist interval, an RLL code with $d + 1 = M_{\text{Tx}}$ must be applied. That is, the run-length encoding limits the interference between the transmit pulses. Due to the 1-bit quantization we only observe the interference in the zero crossings and thus will refer to it as inter zero crossing interference (IZI).

Fig. 3 depicts an example for the noiseless receive signal $s(t)$ and the sample vector $\mathbf{y}$ with and without oversampling w.r.t. the signaling rate. The samples are taken for the case that $\alpha = 1$ and $\epsilon = 0$. We observe that the zero crossings of $s(t)$ and $x(t)$ are almost identical, i.e., for $d + 1 = M_{\text{Tx}}$ and $\alpha = 1$ the IZI can be neglected. This is consistent with [5] where studies of an auxiliary channel law for a bandlimited channel with 1-bit output quantization have shown that IZI can be neglected for $\alpha = 1$. For the case of $\alpha = 0$, i.e., an ideal low pass filter, the rippling of the transmit pulses decays much slower than for $\alpha = 1$ and as a consequence the IZI is much more pronounced.

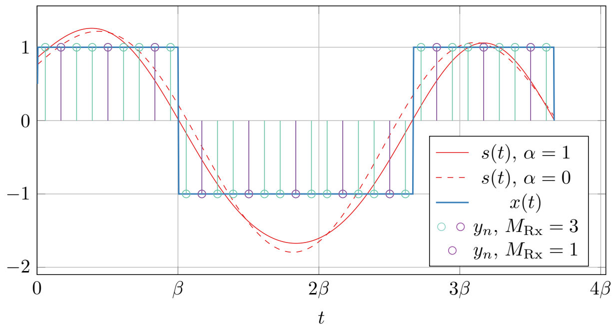

Fig. 3: Example for a noiseless receive signal $r(t) = s(t)$ with signaling factor $M_{\mathrm{Tx}} = 3$ and $\epsilon = 0$

As depicted in Fig. 3 the optimal sampling of $r(t)$, i.e., $\epsilon = 0$, is such that the zero crossings of $x(t)$ are exactly in the middle of two samples. If the transmission is noise and IZI free, a non-zero time shift $\epsilon \in (-0.5, 0.5)$ does not affect $\mathbf{y}_k$. If $r(t)$ is corrupted by noise and IZI, a shift in the sampling grid away from the optimal sampling time instants means that the samples that are shifted towards the zero crossings of $x(t)$ are more sensitive to noise and IZI. Thus, the worst possible time shift is half the time between two samples, i.e., $\epsilon = \pm 0.5$.

Since the elements in $\mathbf{y}$ are not independent (due to RLL encoding, overlapping transmit pulses and colored noise samples), the maximum likelihood sequence estimator (MLSE) is needed to achieve the optimum detection quality in terms of frame error rate [16]. To derive the MLSE, an exact analytical description of the likelihood function $p(\mathbf{y}|\mathbf{x})$ is required. Unfortunately, it is a mathematically open problem to find an analytical description for the likelihood function of system models with correlated Gaussian noise and 1-bit quantization, since there is no analytical description of the orthant probabilities [17].

Hence, we consider the IZI as a noise source and the receiver assumes a memoryless binary symmetric channel (BSC). The MLSE only considers the run-length constraint and thus can be implemented by the Viterbi algorithm with the Hamming distance as metric. To minimize the IZI, we chose $d + 1 = M_{\mathrm{Tx}}$ and $\alpha = 1$. Since this work is in the context of board-to-board computer communication, the resulting out of band power is permissible. The difference to the BSC is due to the fact that for $M_{\mathrm{Tx}} > 1$ elements of $\mathbf{x}$ that are close to the zero crossings are more sensitive to noise, since they are placed inside the transition regions of $s(t)$. Furthermore, the approximation as BSC neglects the noise correlation.

The spectral efficiency is defined as

$$\zeta = \frac{\frac{M_{\mathrm{Tx}}}{\beta} R}{2W} = M_{\mathrm{Tx}} R = (d+1)R \,, \tag{9}$$

where R is the rate of the RLL code. For RLL sequences with maximum entropy rate the spectral efficiency is given in Table 1. Although the entropy rate is decreasing with increasing d, the spectral efficiency is increasing due to higher signaling rates. For our RLL block code with $d = 2$ and $R = 1/2$, the spectral efficiency is $\zeta = 1.5 \frac{\mathrm{bit/s}}{\mathrm{Hz}}$. Note that the roll-off factor does not influence the spectral efficiency, since we defined the bandwidth in terms of the cut-off frequency.

Table 1: Maximum entropy rate and spectral efficiency

d	0	1	2	3	4
$C(d)$	1	0.6942	0.5515	0.4650	0.4057
$\zeta(d)$	1	1.3884	1.6545	1.8600	2.0285

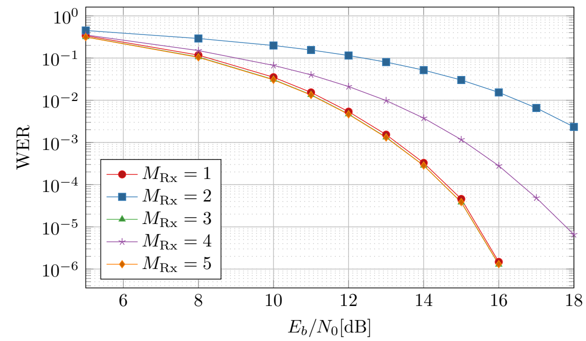

Fig. 4: Effect of the factor M_{Rx} on the WER if $\epsilon = 0$

5. THE EFFECT OF OVERSAMPLING WITH RESPECT TO THE SIGNALING RATE

In this section we will show that oversampling w.r.t. the signaling rate, i.e., $M_{\mathrm{Rx}} > 1$, leads to a vanishing influence of ϵ on the word error rate (WER). A word error occurs when an RLL codeword is demapped into a wrong information word. Since the bit error rate (BER) depends on the specific mapping of the utilized block code between information bits and codewords, the WER is a better suited error measure. We start with considering the system with perfect timing, i.e., $\epsilon = 0$. As depicted in Fig. 4 oversampling w.r.t. the signaling rate has little positive influence on the WER if M_{Rx} is odd and a considerably bad effect if M_{Rx} is even.

To explain this phenomenon, we start with the assumption that an error occurred but the run-length constraint was not violated. In this case oversampling w.r.t. the signaling rate is similar to using a repetition code since there are M_{Rx} samples for every symbol, i.e., instead of only one sample one takes M_{Rx} samples in the region of the symbol. The difference from actually applying a repetition code is that for the symbols next to the zero crossings of $x(t)$, the additional samples are even closer to the zero crossings (see Fig. 3) and are thus more likely to flip. Hence, for odd M_{Rx} the performance gain is probably rather small, in fact there is none. If M_{Rx} is even, there is always the possibility of an irresolute situation where the decision for plus and minus one is equally likely. Hence, a random decision gives a 50% chance of making the correct decision. Since for $M_{\mathrm{Rx}} > 1$ additional samples are closer to the zero crossings of $x(t)$, an irresolute situation due to a zero crossing shift appears more likely than an error in the case of $M_{\mathrm{Rx}} = 1$. Hence, we restrict all further discussions to odd M_{Rx}.

On the other hand, if we assume that the run-length constraint of the sequences was violated, oversampling w.r.t. the signaling rate actually helps the MLSE to recover the correct sequence. However, since RLL codes do not increase the minimum distance [18], this has almost no influence on the WER,

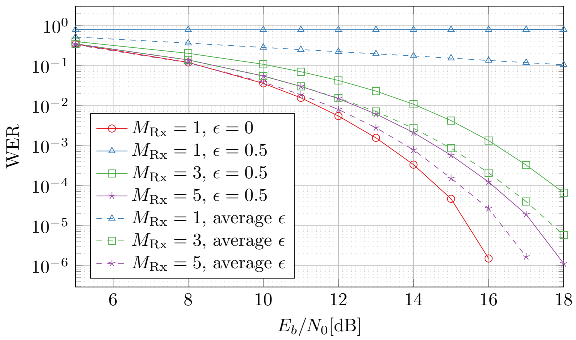

Fig. 5: Effect of the factor M_{Rx} on the WER if $\epsilon \neq 0$

as can be seen in Fig. 4.

If $\epsilon \neq 0$ oversampling w.r.t. the signaling rate is beneficial even if the run-length constraint is not violated. To illustrate this, consider the worst possible time shift $\epsilon = 0.5$. After quantization, the signs of the samples that are placed directly on the zero crossings of $x(t)$ are completely random. This is also true if $M_{\mathrm{Rx}} = 3, 5, 7, \ldots$ but now there are $M_{\mathrm{Rx}} - 1$ additional samples available to recover the correct sequence.

Fig. 3 depicts the transmit symbols and the additional samples for $M_{\mathrm{Rx}} = 3$ and $\epsilon = 0$. Detection is successful if a zero crossing shift due to noise and IZI does not exceed beyond the optimal sampling time instant of the transmit symbol next to the zero crossing. Let us now quantify the range of a zero crossing shift that does not lead to a detection error. For $\epsilon = 0$, a zero crossing in $x(t)$ is $\frac{T_s}{2}$ away from the closest sample. This sample is $\frac{M_{\mathrm{Rx}}-1}{2} T_s$ away from the optimal sampling time instant of the transmit symbol. Now consider that $\epsilon \neq 0$ and recall that $T_s = \frac{\beta}{M_{\mathrm{Tx}} M_{\mathrm{Rx}}}$. The range of a correctable zero crossing shift s is given by

$$\begin{aligned} & s > -\left(\frac{1}{2} + \frac{M_{\mathrm{Rx}} - 1}{2} - \epsilon\right) \frac{\beta}{M_{\mathrm{Tx}} M_{\mathrm{Rx}}} \\ \wedge \quad & s < \left(\frac{1}{2} + \frac{M_{\mathrm{Rx}} - 1}{2} + \epsilon\right) \frac{\beta}{M_{\mathrm{Tx}} M_{\mathrm{Rx}}} . \end{aligned} \tag{10}$$

If $M_{\mathrm{Rx}} = 1$ and $\epsilon = 0$, the range reduces to $-\frac{\beta}{2M_{\mathrm{Tx}}} < s < \frac{\beta}{2M_{\mathrm{Tx}}}$. The same is true if $\epsilon \neq 0$ and $M_{\mathrm{Rx}} \to \infty$. Hence, the additional errors due to $\epsilon \neq 0$ vanish if $M_{\mathrm{Rx}} \to \infty$. This fact is somehow obvious, since the problem of finding the optimal sampling points becomes obsolete if the sampling rate is infinitely large.

Fig. 5 shows the performance improvement caused by oversampling w.r.t. the signaling rate if $\epsilon \neq 0$. If $\epsilon = 0.5$ and $M_{\mathrm{Rx}} = 1$, the WER increases significantly. However, by increasing M_{Rx} the WER converges to the WER of $\epsilon = 0$ and $M_{\mathrm{Rx}} = 1$. This coincides with the above discussions. Hence, we are able to compensate the errors introduced by a time shift ϵ without estimating and correcting ϵ.

Moreover, Fig. 5 depicts the performance for an ϵ that was chosen independently and randomly from $[-0.5, 0.5]$ for every simulation run. This average performance is better than the performance of the worst case $\epsilon = 0.5$, as for small M_{Rx} the actual performance depends strongly on the specific ϵ with the worst case $\epsilon = \pm 0.5$ and the best case $\epsilon = 0$. One way to circumvent this dependency of the specific ϵ, is to apply

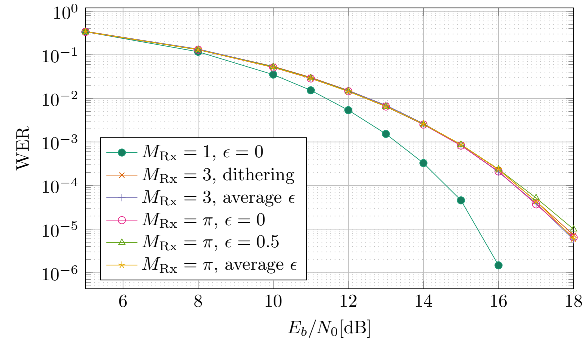

Fig. 6: Effect of dithered and irregular sampling

dithered sampling, i.e.,

$$r_n = r\left(nT_s + \frac{T_s}{2} + \epsilon_n T_s\right), \; \epsilon_n \sim \mathcal{U}\left(-0.5, 0.5\right), \tag{11}$$

with independently drawn ϵ_n. Hence, instead of equidistant sampling, the samples are taken at random time instances without crossing the interval to the next sample. This results in a randomized sampling grid and a performance equal to the average performance with a fixed ϵ, as shown in Fig. 6.

As dithered sampling is difficult to implement and integer oversampling can in general not be guaranteed, we now consider an irrational oversampling factor M_{Rx}. The sampling grid will be irregular w.r.t. the optimal sampling time instants, and thus the performance will be independent of ϵ. Fig. 6 depicts this effect for $M_{\mathrm{Rx}} = \pi$. On the other hand, if M_{Rx} is not an integer, the receive vector $\mathbf{y}_k$ can have either $\lceil M_{\mathrm{Rx}} \rceil$ or $\lfloor M_{\mathrm{Rx}} \rfloor$ elements. In order to resolve which samples belong to which transmit symbol x_k, one must know the time shift ϵ. Since estimating ϵ on 1-bit quantized samples is an open problem, we considered perfect knowledge in the simulations. The study of this estimation problem remains for future work. Obviously, drastically increasing M_{Rx} would compensate for not knowing ϵ but is probably more costly than estimating ϵ.

6. CONCLUSION

We study an RLL sequence based communication system with 1-bit quantization and FTN signaling. Additional oversampling w.r.t. the signaling rate is performed at the receiver. The system is considered to be perfectly synchronized, except for a fixed but unknown time shift. We observed that for small oversampling rate the WER is dependent on the actual time shift. This can be circumvented by dithered sampling, which results in a randomized sampling grid. The same effect can be achieved with an irrational oversampling factor. On the downside, an irrational oversampling factor requires knowledge of the time shift. Its estimation remains for future work. However, with increasing oversampling rate, the detection error due to a fixed time shift vanishes without estimating and correcting the time shift.

7. ACKNOWLEDGMENT

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