An Information Theoretic Analysis of Sequential Decision-Making

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Abstract—We provide a novel analysis of Wald’s sequential probability ratio test based on information theoretic measures. This test is optimal in the sense that it yields the minimum mean decision time. To analyze the decision-making process we consider information densities enabling to represent the stochastic information content of the observations yielding a stochastic termination time of the test. We study the consequences of the optimality of the test on the change of the information density in time. As a corollary, we find that the optimality of the test implies that the conditional probability to decide for hypothesis \( H_1 \) (or the counter-hypothesis \( H_0 \)) given that the test terminates at time instant \( k \) is independent of time. Moreover, we evaluate the evolution of the mutual information between the binary variable to be tested and the decision variable of the Wald test.

I. INTRODUCTION

We provide an information theoretic analysis of sequential decision-making. In many decision problems it is important to make decisions as fast as possible and at the same time with a given reliability. In this regard, e.g., consider information processing in living organisms. Cells decide about their fate, e.g., if they differentiate, proliferate, or die, based on noisy observations. Reducing decision times can enhance the capability of a microorganism to adapt to the environment [1]. The fundamental trade-off in living organisms between the reliability of a decision and the time to make a decision can be modeled as an optimization problem in the area of decision theory. Information theory already provides an understanding of some processes in biology [2]. This motivates us to study sequential decision-making by information theoretic measures.

Mathematically, the above problem has been first studied in the seminal work by A. Wald who introduced a sequential probability ratio test to enable fast decisions between two possible hypotheses [3]. For independent and identically distributed observations this test yields the minimum mean decision time for a decision with a given probability of error [4]. The test accumulates the likelihood ratio given by the sequence of observations and decides as soon as this cumulative likelihood ratio reaches a given threshold which depends on the required reliability of the decision. The decision time is a random variable that depends on the actual observation sequence. Wald’s test is an important application of an erasure test where the observer can decide to take additional observations. Since there is cost associated with the observation process it is desirable to minimize the number of observations. The Wald sequential test minimizes the expected cost at every time instant \( k \) [5, Ch. 6].

While Wald’s test – and more general sequential decision-making – has found widespread application, there is to the best of our knowledge no analysis of the test in terms of information theoretic measures. However, to understand decision-making in information processing systems like biological systems an information theoretic understanding of sequential decision-making is crucial. In this regard, several questions arise: 1) What are suitable information measures that enable to characterize the sequential decision-making process, especially with respect to the fact that the termination time of the test depends on actual realizations of the observation processes and not on process averages? 2) How is the optimality of the Wald test, which leads to the minimum mean decision time, reflected in terms of relations between information measures? 3) How does the mutual information between the binary variable to be tested and the decision variable of the Wald test evolve during the decision process?

The present paper tries to shed some light on the preceding questions. For a binary sequential hypothesis testing problem we show that the information density is an appropriate measure to account for the dependence of termination time of the test on the observed sequence. Mutual information falls short due to its inherent nature of describing the behavior of sample averages over all processes. We derive a recursive expression for the information density describing the behavior of the Wald test. This recursive relation describes the optimality of the test with respect to the decision time in terms of information theoretic quantities — thus connecting decision theory and information theory. This is one of the main contributions of this paper. Moreover, based on the recursive expression for the information density, we show that a key

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property of the Wald test is that the conditional probability to decide for hypothesis $H_1$ (or the counter-hypothesis $H_0$) given that the test terminates at time instant $k$ is independent of time. The same independence of the decision probabilities was previously found via an entirely different route for a continuous-time scenario. Mathematically, the Wald test is analogous to a continuous-time first-passage problem with two absorbing boundaries. For a system described by a nonlinear stochastic differential equation of the Langevin type it has been shown that the probability to be terminated at one of the two boundaries is independent of the time of absorption [6], [7]. In addition, we provide an expression characterizing the evolution of the mutual information between the binary random variable to be tested and the decision variable of the Wald test.

II. SYSTEM MODEL

We consider the following decision problem on the binary random variable $X \in \{-1, 1\}$ based on a sequence of noisy observations of $X$ where the $k$th observation is given by

$$Y_k = \sqrt{\rho} X + Z_k, \quad k \in \mathbb{N}.$$  \hspace{1cm} (1)

Here, $Z_k$ is additive noise with zero mean and unit variance. The individual noise samples are assumed to be independent identically distributed (i.i.d.) with density $p_Z$. In addition, we assume that the noise distribution is symmetric with respect to zero, i.e., $p_Z(-z) = p_Z(z)$ which is for example fulfilled by a zero-mean Gaussian distribution. Moreover, $\rho$ is a constant which can be interpreted as the signal-to-noise ratio. We assume that both hypothesis are equally likely, $P(X = 1) = P(X = -1) = \frac{1}{2}$.

The aim is to decide as fast as possible, i.e., with the lowest possible number of observations $Y_k$, if $X = 1$ (hypothesis $H_1$) or if $X = -1$ (hypothesis $H_0$) with a given reliability. In his seminal paper [3] Wald solved this problem by providing a sequential probability ratio test (the Wald test), which is optimal in the sense that it minimizes the mean decision time for a given reliability. The decision time is itself a random variable, which depends on the actual realization of the observation sequence. For this purpose, the Wald test collects observations $Y_k$ until the cumulated log-likelihood ratio

$$S_k = \sum_{l=1}^{k} L_k = \sum_{l=1}^{k} \log \left( \frac{p_{Y_k|X=1}}{p_{Y_k|X=-1}} \right)$$ \hspace{1cm} (2)

exceeds (falls below) a prescribed threshold $T_1$ ($T_0$). The test decides for $H_1$ ($H_0$) when $S_k$ first crosses $T_1$ ($T_0$). Here, $p_{Y_k|X}$ denotes the probability density function of the observations $Y_k$ conditioned on the event $X$. The thresholds $T_1$ and $T_0$ depend on the maximum allowed probabilities for making a wrong decision $\alpha_1 = P(D = 1|X = -1)$ and $\alpha_0 = P(D = -1|X = 1)$ where $D \in \{-1, 1\}$ is the decision of the test. Here, $P(D = 1|X = -1)$ ($P(D = -1|X = 1)$) denotes the probability that the test decides for hypothesis $H_1$ ($H_0$) although $H_0$ ($H_1$) is true. The thresholds $T_1$ and $T_0$ are functions of the maximum allowed error probabilities $\alpha_1$ and $\alpha_0$. As their determination is rather involved, in the following we use the approximations

$$T_0 = \log \frac{\alpha_1}{1 - \alpha_0}, \quad T_1 = \log \frac{1 - \alpha_1}{\alpha_0}$$ \hspace{1cm} (3)

with $\alpha_0, \alpha_1 < 0.5$. This choice still guarantees that the actual error probabilities are not larger than the maximum allowed error probabilities. If the test terminates exactly on one of the thresholds, the actual error probabilities $P(D = 1|X = -1)$ and $P(D = -1|X = 1)$ coincide with the maximum allowed error probabilities $\alpha_1$ and $\alpha_0$. If the mean and the variance of the increments $L_k$ are small in comparison to the thresholds the test ends close to one of the thresholds [3, pp. 132-133], which we assume in the following. Moreover, we assume $\alpha_0 = \alpha_1 = \alpha$. Hence, we can define $T = T_1 = T_0$.

For the analysis of the Wald test we introduce the model in Fig. 1. Here, the ternary variable $U_k \in \{-1, \epsilon, 1\}$ with

$$U_k = \begin{cases} 1 & \text{if } \exists S_l \geq T \text{ for } l \leq k, \\ -1 & \text{if } \exists S_l \leq -T \text{ for } l \leq k, \\ \epsilon & \text{otherwise} \end{cases} \hspace{1cm} (4)$$

and $U_0 = \epsilon$ describes the state of the Wald test. The state $U_k = 1$ (−1) in case the cumulative log-likelihood value $S_k$ has reached the threshold $T$ (−$T$) at any time instant up to the observation time instant $k$. In case $S_k$ has not yet reached any of the two thresholds $U_k$ equals $\epsilon$ where $\epsilon$ denotes an erasure corresponding to the case that a decision with the required reliability is not possible based on the first $k$ observations. As soon as $U_k$ becomes 1 or −1 the actual test terminates with the corresponding decision $D$, otherwise it will take an additional observation into account. In our analysis, we assume that no termination occurs, but we just assume that once $U_k = a$ with $a \in \{-1, 1\}$ it remains in this state. The evolution of the state variable $U_k$ can also be described by the trellis and the state transition diagram in Fig. 2. The variable $D \in \{-1, 1\}$ corresponds to the actual decision of the Wald test with $D = a$ if $U_k = a$ and otherwise $D$ is undefined.

A. Stochastic Characterization

To analyze the behavior of the Wald test we derive relations for the probability distribution of the state variables $U_k$. In the following, we assume without loss of generality that hypothesis $X = 1$ is true. This is possible as we consider symmetric thresholds and a symmetric noise distribution yielding

$$P(U_k = 1|X = -1) = P(U_k = -1|X = 1)$$ \hspace{1cm} (5)

$$P(U_k = 1|X = 1) = P(U_k = -1|X = -1)$$ \hspace{1cm} (6)
The probability that the Wald test has already made a decision for the hypothesis $H_1$ ($H_0$) corresponding to $X = 1$ ($X = -1$) at the time instant $k$ or before can be expressed as

$$P(U_k = a|X = 1) = P(U_{k-1} = a|X = 1) + P(U_k = a|U_{k-1} = \epsilon, X = 1)P(U_{k-1} = \epsilon).$$  

(7)

with $a \in \{1, -1\}$. For (7) we have used that

$$P(U_{k-1} = \epsilon|X = 1) = P(U_{k-1} = \epsilon) \quad (8)$$

which follows from

$$P(U_{k-1} = \epsilon) = \frac{1}{2} \{P(U_{k-1} = \epsilon|X = 1) + P(U_{k-1} = \epsilon|X = -1)\}$$

(9)

and as we assume that both events $X = 1$ and $X = -1$ are equally likely. Additionally (9) follows from the assumption of a symmetric noise distribution $p_\epsilon$ with zero mean. Thus, (7) provides a recursive relation for $P(U_k = a|X = 1)$. The initial distribution is given by $P(U_0 = \epsilon) = 1$ as at time instant $k = 0$ no observation has been considered and the Wald test has not yet decided for one of the hypothesis. Note that with (7)

$$P(U_k = a|X = 1) = \sum_{l=1}^{k} P(U_l = a|U_{l-1} = \epsilon, X = 1)P(U_{l-1} = \epsilon)$$

as $P(U_0 = a|X = 1) = 0$.

Note that $P(U_k = a|X = 1)$ corresponds to the probability that the Wald test has terminated at the positive (negative) threshold at any time instant up to the time instant $k$. On the other hand, $P(U_k = a|U_{k-1} = \epsilon, X = 1)$ is the probability that the Wald test terminates at the positive (negative) threshold at the time instant $k$. Fig. 3 shows an example of the evolution of these probability distributions over time $k$.

Finally, due to the assumption of symmetric thresholds, symmetric noise, and the assumption that $X = 1$ and $X = -1$ are equally likely, it can be shown that for all $k$

$$P(U_k = 1) = P(U_k = -1) \quad (10)$$

$$P(U_k = 1|U_{k-1} = \epsilon) = P(U_k = -1|U_{k-1} = \epsilon). \quad (11)$$

### III. Analysis by Information Measures

#### A. Optimality of Sequential Decision-Making

One of the motivations of the present work is to study the behavior of such a sequential probability ratio test based on information measures. Although the mutual information is not the appropriate measure, for ease of explanation we will start by it before shifting to information density.

![Trellis (a) and state transition diagram (b) of $U_k$](image)

![Termination probability at the time instant $k$](image)

![Termination probability up to the time instant $k$](image)
The mutual information between $X$ and the whole sequence of decision variables $\mathbf{U}^k$ can be expressed as a sum of the increments of the mutual information with each time step.

The Wald test leads to the minimum mean decision time. What does this imply in terms of the mutual information? The increase in mutual information between the sequence $\mathbf{U}^k$ and $X$ from the time $k-1$ to time $k$ is given by

$$I(X; \mathbf{U}^k) - I(X; \mathbf{U}^{k-1}) = I(X; U_k | \mathbf{U}^{k-1})$$

where we have used the chain rule for mutual information and where (20) follows from (13) to (15).

At every time instant $k$ the test adds the log-likelihood ratio $L_k$ of the additional observation to $S_{k-1}$ and compares the result with the thresholds (add-compare-select). The decision variable $U_k$ only provides the information if $S_k$ has reached or crossed one of the levels given by the threshold $T$ or $-T$ (corresponding to a specific probability of $X$ being equal to 1 or $-1$) or if it is between both. In case one of the thresholds has been reached, the test terminates and no further observations are taken such that the probability of $X$ conditioned on the decision variables does not change anymore. Thus, by construction of the test the probability of $X$ conditioned on the decision variables $\mathbf{U}^k$ solely depends on the last decision, i.e.,

$$P(X | \mathbf{U}^k) = P(X | U_k).$$

Using Bayes’ rule and (21) we obtain

$$P(X, U_{k-1} | U_k) = P(X | U_k) P(U_{k-1} | U_k).$$

Hence, $X$ and $U_{k-1}$ are conditionally independent. The following Markov chain holds $X \leftrightarrow U_k \leftrightarrow U_{k-1}$. Therefore

$$I(X; U_{k-1} | U_k) = 0.$$  

(23)

$U_k$ carries all information on $X$ that is contained in $U_{k-1}$.

With the chain rule and (23) we can express (20) by

$$I(X; U_k | U_{k-1}) = I(X; U_k) - I(X; U_{k-1})$$

$$\Leftrightarrow I(X; U_k) = I(X; U_{k-1}) + I(X; U_k | U_{k-1}).$$

(24)

Eq. (24) describes the increase of the mutual information between $X$ and the decision variable $U_k$ over one time step. The Wald test makes the decision based on a comparison of the cumulative log-likelihood ratio $S_k$ with the thresholds. Hence, $S_k$ uniquely determines $U_k$ at the time instant $k$. Moreover, the decision of the Wald test at time instant $k$ (decide for one of the hypothesis or take another observation) solely depends on $U_k$ and not on its whole sequence given by $\mathbf{U}^k$. Thus, $I(X; U_k)$ is the mutual information the test gives on $X$ at time instant $k$. By (24) this information is given by the corresponding mutual information at the preceding time instant plus the incremental information given by (20). Hence, (24) shows how the behavior of the Wald test, which yields a minimum mean decision time, is reflected in the mutual information.

2) Information Density: With (24) we have studied the Wald test based on the mutual information, which reflects the behavior of averages over all observation sequences $Y_1, \ldots, Y_k$. However, the termination behavior of the Wald test depends on the specific realization of the observation sequence and not on the average over all realizations. For each individual sequence the Wald test should terminate as early as possible with a predefined error probability. Thus, we need an expression reflecting this behavior of the Wald test corresponding to (24) but being more restrictive in the sense that it holds for every individual observation process. This is achieved by the following equation based on information densities

$$i(X; U_k) = i(X; U_{k-1}) + i(X; U_k | U_{k-1}).$$

(25)

The information density between $X$ and $U_k$ is defined as $\text{(6)}$

$$i(X; U_k) = \log \left( \frac{P(U_k | X)}{P(U_k)} \right).$$

(26)

Note that the mutual information $I(X; U_k)$ is the expectation of the information density, i.e., $I(X; U_k) = \mathbb{E}_{U_k}[i(X; U_k)]$. Analogous the conditional information density is given by

$$i(X; U_k | U_{k-1}) = \log \left( \frac{P(U_k | U_{k-1}, X)}{P(U_k | U_{k-1})} \right).$$

(27)

Hence, we get (24) by taking the expectation of (25) with respect to all random quantities and, thus, (25) must hold for all combinations of events of $U_k$, $U_{k-1}$, and $X$.

B. Implications of the Optimality

Based on (25) we show that in case the test terminates at time instant $k$ the probability to decide for hypothesis $H_1$, i.e., $P(U_k = 1 | U_{k-1} = \epsilon, X = 1, U_k \neq \epsilon)$, (or the counter-hypothesis $H_0$) is independent of time. It can be shown that this statement is equivalent to the property that the ratio of the termination probabilities at both boundaries is independent of the time instant $k$, i.e.,

$$\frac{P(U_k = 1 | U_{k-1} = \epsilon, X = 1)}{P(U_k = -1 | U_{k-1} = \epsilon, X = 1)} = \kappa \quad \forall k \in \mathbb{N}$$

(28)

where $\kappa$ is a constant which in case we assume that the test terminates exactly on one of the thresholds is equal to $\frac{1-\alpha}{\beta}$.

In the following, we will prove that (28) is a direct consequence of (25), which can be rewritten as

$$\frac{P(U_k | X)}{P(U_{k-1} | X)} = \frac{P(U_{k-1} | X) P(U_k | U_{k-1}, X)}{P(U_{k-1} | X) P(U_k | U_{k-1})}.$$  

(29)

While (29) also directly follows from (22), the expression based on information densities in (25) and the mutual information based version in (24) give an intuitive representation. Note that the Wald test fulfills (29) for all combinations of $X \in \{-1, 1\}$, and $U_k, U_{k-1} \in \mathcal{U}$. Due to the symmetry of the problem we consider only the case $X = 1$ w.l.o.g.. Evaluating (29) for different values of $U_k$ and $U_{k-1}$ yields...
For (30) to (33) we have used that $P(U_k = 0 | U_{k-1} = 0, X = 1) = 1$ and $P(U_k = 0 | U_{k-1} = 0) = 1$ with $a \in \{1, -1\}$ and (8).

Due to the symmetry of the test and the considered scenario reflected by $10$, based on (30) and (33) we get

\[
\frac{P(U_k = 1 | U_{k-1} = 1)}{P(U_k = 1)} = \frac{P(U_k = 1 | U_{k-1} = 1)}{P(U_{k-1} = 1)} = \kappa. \tag{34}
\]

I.e., the ratio between the probability that the Wald test terminates at the positive boundary and the probability that it terminates at the negative boundary at any time instant up to the time instant $k$ is constant over $k$. We denote this constant as $\kappa$. Using (31), (32), (10), (11), and (34) we get

\[
\frac{P(U_k = 1 | X = 1)}{P(U_k = -1 | X = 1)} = \frac{P(U_k = 1 | U_{k-1} = 1)}{P(U_{k-1} = 1)} = \kappa. \tag{35}
\]

We denote this constant independent of the time instant $k$ which proves (35).

It can be shown that the Wald test terminates with probability one [5] Th. 6.2-1 which means that $\lim_{k \to \infty} P(U_k = 1 | U_{k-1} = 1) = 0$. Hence, in case we assume that the test terminates exactly at one of the thresholds, i.e., $S_k = \pm T$, by the design of the test it holds that

\[
\lim_{k \to \infty} P(U_k = -1 | X = 1) = \alpha \tag{36}
\]

\[
\lim_{k \to \infty} P(U_k = 1 | X = 1) = 1 - \alpha \tag{37}
\]

yielding with (35) $\kappa = \frac{1-\alpha}{\alpha}$.

Thus, we have shown that the optimality of the Wald test to decide at the earliest possible time instant for a given reliability, which is reflected by (25), implies that the ratio of the termination probabilities on both thresholds is independent of the time $k$. To the best of our knowledge, this result is new.

This behavior of the Wald sequential test, which is discrete-time, is similar to a continuous-time first passage problem with two absorbing boundaries. E.g., for a system described by a nonlinear stochastic differential equation of the Langevin type this continuous-time first-passage problem shows also the behavior that the ratio of the termination probabilities at both boundaries is time independent, see [7], [6].

C. Evolution of Mutual Information

At time $k$ the mutual information between $X$ and the decision variable $U_k$ of the Wald test is given by

\[
I(X; U_k) = P(U_k = 1 | X = 1) + P(U_k = -1 | X = 1) \times \log \left( \frac{P(U_k = 1 | X = 1)}{P(U_k = -1 | X = 1)} \right) + P(U_k = 1 | X = 1) \log \left( \frac{P(U_k = 1 | X = 1)}{P(U_k = -1 | X = 1)} \right) \tag{38}
\]

Under the assumption that the test terminates exactly on one of the thresholds with (35) and $\kappa = \frac{1-\alpha}{\alpha}$ we get

\[
I(X; U_k) = \frac{P(U_k = 1 | X = 1)}{1 - \alpha} I(X; U_\infty). \tag{39}
\]

I.e., $I(X; U_k)$ linearly increases with $P(U_k = 1 | X = 1)$ until it achieves the final value, cf. (37)

\[
I(X; U_\infty) = 1 + \alpha \log(\alpha) + (1 - \alpha) \log(1 - \alpha) \tag{40}
\]

which corresponds to the mutual information to be achieved to allow a decision with the predefined error probability.

IV. CONCLUSION

We have provided a detailed study of Wald’s sequential test based on information theoretic measures. The information density is an appropriate measure to reflect the property of the Wald sequential test that the decision time and result depend on the individual realization of the observations. We have derived an equation describing the behavior of the Wald test in terms of information densities. Based on this we have shown that in case the test terminates at time instant $k$ the probability to decide for hypothesis $H_1$ (or the counter-hypothesis $H_0$) is independent of time. An analogous property has been found for a continuous-time first passage problem with two absorbing boundaries in the context of non-equilibrium statistical physics. Finally, we have evaluated the evolution of the mutual information between $X$ and the decision variable $U_k$ with time $k$. Generalizations of the present work, e.g., to non-equally likely events $X = 1$ and $X = -1$ remain for further work.

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