2nd part of the lecture: Elasticity theory and beyond

- basics: strain, stress, elastic energy, governing equations
- basic examples
- Pappkovich-Neuber approach
- Green's function: Thomson, Boussinesq
- beam theory
- elastic plates and shells
- Contact mechanics
- Eshelby theory (Benj.)
- viscoelasticity and poroelasticity
- stresses in tissues
- cell rheology: methods and basic results

5 lectures left
$\longrightarrow$ choose topics

What is a

fluid?

- no long range molec. order
- no shear elasticity

↪ no shape memory

⇕

hydrodynamics

solid

- long-range molecular order
- shear elasticity

↪ shape memory

⇕

elasticity theory

Elasticity theory

I. Introduction

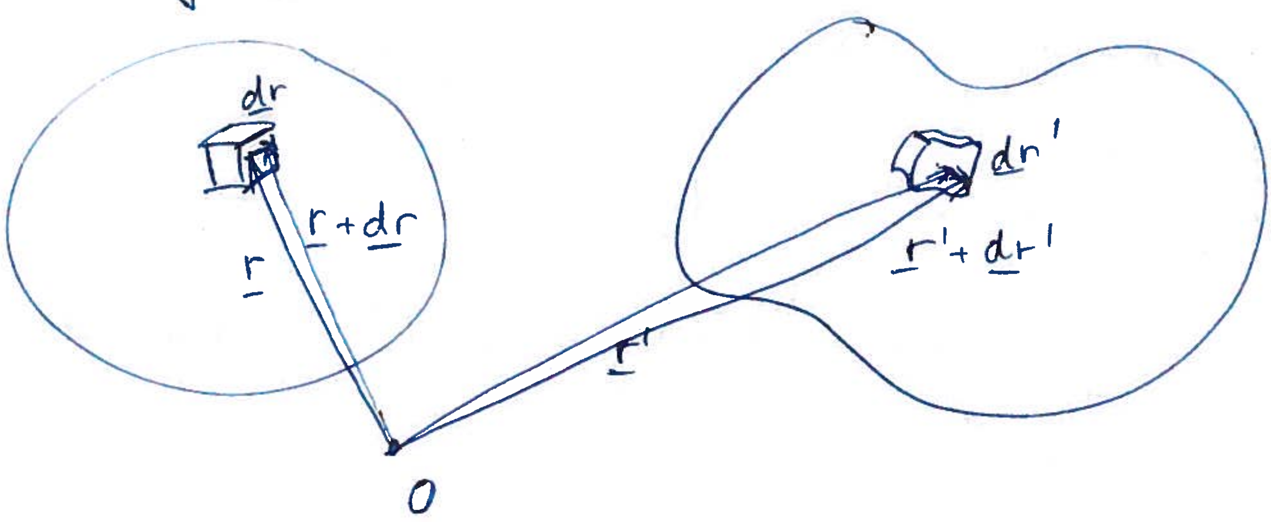
I.1. General solid bodies (rubber, steel, ...)

- shape memory, tend to return to their original shape / configuration after external forces are switched off
  $\equiv$ elastic
- contrasts to plastic behaviour
  $\equiv$ ext. forces cause irreversable deformation

- elastostatics
  $\equiv$ study of elastic deformations under condition of force equilibrium
  $\rightarrow$ no time changes, no movements
  $\rightarrow$ will focus on this

Literature: Landau & Lifshitz
Audoly, Pomeau "Elasticity & Geometry"
A. Lurie: "Theory of Elasticity"

I.2. Deformation of a solid via external forces



- point at position $\underline{r}$ is ~~translocated~~ to $\underline{r}'(\underline{r})$

$$\underline{r} \rightarrow \underline{r}'(\underline{r})$$

displacement field

$$\underline{u}(\underline{r}) := \underline{r}'(\underline{r}) - \underline{r}$$

- changes of distance between points
  angle between points

Distance change

$$d\underline{r}'^2 = d(\underline{r} + \underline{u})^2 = \left(dr_i + \frac{\partial u_i}{\partial r_j} dr_j\right)^2$$

$$= d\underline{r}^2 + 2\frac{\partial u_i}{\partial r_j} dr_i dr_j + \frac{\partial u_i}{\partial r_j}\frac{\partial u_i}{\partial r_k} dr_j dr_k$$

$$= d\underline{r}^2 + \left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) dr_i dr_j + \frac{\partial u_i}{\partial r_j}\frac{\partial u_i}{\partial r_k} dr_j dr_k$$

I.3. Introduce the strain tensor

$$\varepsilon_{ij} := \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) + \frac{1}{2}\frac{\partial u_k}{\partial r_i}\frac{\partial u_k}{\partial r_j}$$

we have

$$d\underline{r}'^2 - d\underline{r}^2 = 2\,\varepsilon_{ij}\, dr_i dr_j$$

- strain rate from hydrodynamics Γ is time-derivati

$$\Gamma = \dot{\varepsilon}$$

Approximation of small displacement:

- assuming $u(\underline{r})$ is small, strain can be linearised

$$\varepsilon_{ij}(\underline{r}) \approx \frac{1}{2}\left(\frac{\partial u_i}{\partial r_j} + \frac{\partial u_j}{\partial r_i}\right) + O(u^2)$$

- full-body translations or rotations can be disregarded in displacement, only transformations matter that change distance between points in the body

⟶ adopt this approximation now

- principal strain directions
  - ⟶ strain tensor can be diagonalized by rotation of coord. system

$$\hat{\varepsilon}_{ij} = \begin{pmatrix} \hat{\varepsilon}_{xx} & 0 & 0 \\ 0 & \hat{\varepsilon}_{yy} & 0 \\ 0 & 0 & \hat{\varepsilon}_{zz} \end{pmatrix}$$

$\boxed{\text{example}}$ stretch along x-direction combined with x-y rotation by angle ϕ

$$x' = \lambda \cdot x \cos\varphi - y \sin\varphi$$
$$y' = \lambda x \sin\varphi + y \cos\varphi$$
$$z' = z$$

displacement $\underline{u} = \begin{pmatrix} x(\lambda \cos\varphi - 1) - y \sin\varphi \\ \lambda x \sin\varphi + y(\cos\varphi - 1) \\ 0 \end{pmatrix}$

$\epsilon_{ij} = 0$ except for ϵ_{xx}

$$\epsilon_{xx} = \underbrace{\frac{\partial u_x}{\partial x}}_{\lambda\cos\varphi - 1} + \frac{1}{2}\left(\underbrace{\left(\frac{\partial u_x}{\partial x}\right)^2}_{\lambda^2\cos^2 + 1 - 2\lambda\cos\varphi} + \underbrace{\left(\frac{\partial u_y}{\partial x}\right)^2}_{\lambda^2\sin^2\varphi} + \underbrace{\left(\frac{\partial u_z}{\partial x}\right)^2}_{0} \right)$$

$$= \cancel{\lambda\cos\varphi} - 1 + \frac{\lambda^2}{2} + \frac{1}{2} - \cancel{\lambda\cos\varphi}$$

$$= \frac{\lambda^2 - 1}{2}$$

→ strain removes rigid body rotation after stretching

→ this relies on quadratic terms in strain tensor for arbitrary deformations in this example

- stretching of length element in x-direction

$$\frac{|dr_x'|}{|dr_x|} = \frac{\sqrt{dx^2 + 2\hat{\epsilon}_{xx} dx^2}}{\sqrt{dx^2}}$$

$$= \sqrt{1 + 2\hat{\epsilon}_{xx}}$$

$$\hat{\approx} \quad 1 + \hat{\epsilon}_{xx}$$

small displ., strain

- Each axis of principal strain is stretched by a factor $\lambda_i = 1 + \epsilon_{ii}$

- Volume change

$$dV' = dx' dy' dz'$$

$$= (1 + \hat{\epsilon}_{xx}) dx \, (1 + \hat{\epsilon}_{yy}) dy \, (1 + \hat{\epsilon}_{zz}) dz$$

small strain

$$\cong (1 + \hat{\epsilon}_{xx} + \hat{\epsilon}_{yy} + \hat{\epsilon}_{zz}) \, dx \cdot dy \, dz + O(\epsilon^2)$$

$$= (1 + \mathrm{tr}\,\epsilon) \, dV$$

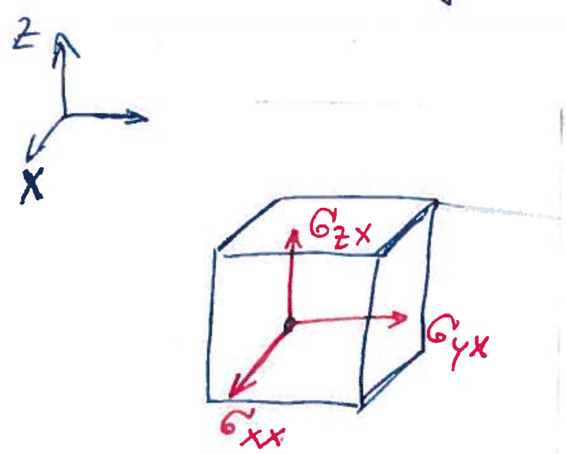
relative volume change $\frac{dV' - dV}{dV} = \mathrm{tr}\,\epsilon$

4 Stress tensor

- when bodies get deformed, arrangement of molecules/atoms gets out of equilibrium
  ↪ internal, restalling forces ≡ internal stresses
  * short-ranged (not like gravity or electr. fields)
  * force on portion of body is the sum of forces acting on subvolumes

$$\underline{F}_{tot} = \int \underline{f}\, dV$$

* the exterior of a volume element exerts forces via contact surfaces
  → force can be written as surface integral



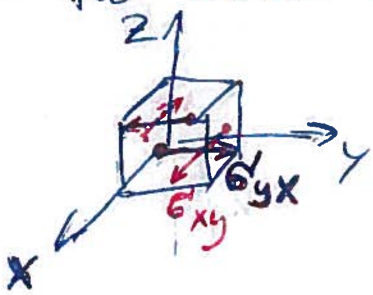
$$\underline{F}_{tot} = \int \operatorname{div} \underline{\underline{\sigma}} \cdot dV = \int \frac{\partial}{\partial r_k} \sigma_{ik}\, dV = \int_{\partial V} \sigma_{ik}\, dS_k$$

$$\partial V$$

$$\boxed{f_i = \frac{\partial}{\partial r_k} \sigma_{ik}}$$

- $\begin{pmatrix} \sigma_{xx} \\ \sigma_{yx} \\ \sigma_{zx} \end{pmatrix} ds^2$ is the force acting on surface of a cube with surface normal $\underline{e}_x$, side length ds
  → force exerted by the exterior of this volume element onto this volume element

- the stress tensor is (normally) symmetric



torque around z-axis

$$dM_z = \frac{dx}{2}\cdot\sigma'_{yx}\, dy\, dz$$

$$+ \frac{dy}{2}\left(-\sigma'_{xy}\right) dx\, dz$$

$$+ \frac{dx}{2}\left(-\sigma'_{yx}\right) dy\, dz - \frac{dy}{2}\sigma'_{xy}\, dx\, dz$$

$$+ O(dx^4)$$

in equilibrium $dM_z = 0$ (no external volumic torques exerted)

$$\Downarrow$$

$$\sigma'_{yx} = \sigma'_{xy}$$

I.5. Elastic energy of isotropic, homogeneous material

Each strain is sum of a pure shear (shape change, no volume change) and a homogeneous compression/dilation

$$\underline{\underline{\varepsilon}} = \underbrace{\left(\underline{\underline{\varepsilon}} - \tfrac{1}{3}\mathrm{tr}(\varepsilon)\,\mathbb{1}\right)}_{\text{pure shear}} + \underbrace{\tfrac{1}{3}\mathrm{tr}(\varepsilon)\,\mathbb{1}}_{\text{homog. expansion/compr.}}$$

Energy of deformation is a Taylor expansion in strain, up to second order

$$E = E_0 + \tfrac{1}{2} K \varepsilon_{ll}^2 + \mu\left(\varepsilon_{ik} - \tfrac{1}{3}\mathrm{tr}(\varepsilon)\,\delta_{ik}\right)^2$$

K... bulk modulus or modulus of compression

μ... shear modulus
(also G in the literature)

→ Hookean elasticity
(Hookean spring $F = K \frac{x^2}{2}$)

- What is the infinitesimal work associated to infinitesimally deform a body via surface forces

$$\delta W = \int_{\partial V} \underline{f}^{ext} \cdot \delta \underline{u} \, dA = \int_{\partial V} (\underline{\underline{\sigma}} \cdot \underline{n}) \cdot \delta \underline{u} \, dA$$

$$= \int_{\partial V} (\underline{\underline{\sigma}} \, \delta \underline{u}) \cdot d\underline{A}$$

$$= \int_V \mathrm{div}(\underline{\underline{\sigma}} \, \delta \underline{u}) \, dV$$

$$= \int \underbrace{\partial_j \sigma_{ij}}_{=0} \delta u_i + \sigma_{ij} \partial_j \delta u_i \, dV$$

(no interior forces)

$$= \int \sigma_{ij} \delta \varepsilon_{ij} \, dV$$

- Because of the above result for the work associated with displacement, we have to have

$$\sigma_{ij} = \left. \frac{\partial E}{\partial \varepsilon_{ij}} \right|_{T = const.} \Longrightarrow E = \tfrac{1}{2} \sigma_{ij} \varepsilon_{ij}$$

(free energy density)

using

$$E = E_0 + \tfrac{1}{2} K \varepsilon_{ll}^2 + \mu \left(\varepsilon_{ik} - \tfrac{1}{3} \mathrm{tr}(\varepsilon) \delta_{ik} \right)^2$$

$$\Rightarrow dE = K \varepsilon_{ll} \, d\varepsilon_{ll} + 2\mu \left(\varepsilon_{ik} - \tfrac{1}{3} \varepsilon_{ll} \delta_{ik} \right) \cdot d\left(\varepsilon_{ik} - \tfrac{1}{3} \varepsilon_{ll} \delta_{ik} \right)$$

$$= K \varepsilon_{ll}\, d\varepsilon_{ll} + 2\mu \left(\varepsilon_{ik} - \tfrac{1}{3}\varepsilon_{ll}\delta_{ik}\right) d\varepsilon_{ik}$$

$$\rightsquigarrow \quad \sigma_{ik} = K\varepsilon_{ll}\delta_{ik} + 2\mu\left(\varepsilon_{ik} - \tfrac{1}{3}\varepsilon_{ll}\delta_{ik}\right)$$

short calculation gives

$$\varepsilon_{ik} = \delta_{ik}\frac{\sigma_{ll}}{9K} + \left(\sigma_{ik} - \tfrac{1}{3}\sigma_{ll}\delta_{ik}\right)/2\mu$$

Comments:

- Note: $\sigma_{ik} \propto \varepsilon_{ik}$ for $i \neq k$
- pure shear: $\varepsilon_{ll} = 0 \longrightarrow \sigma_{ik} = 2\mu\,\varepsilon_{ik}$
- pure compression / expansion

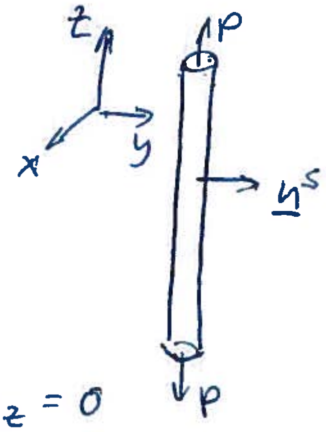
$$\sigma_{ik} = K\varepsilon_{ll}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

I.6 Homogeneous extension of a rod

- forces are applied at the ends of the rod

$\varepsilon_{ik} = \text{const.}$

$\sigma_{ik} = \text{const.}$



$$\underline{\underline{\sigma}} \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = p \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \longrightarrow \sigma_{zz} = p$$

$$\underline{\underline{\sigma}} \cdot \underline{n}^s = 0 \longrightarrow \sigma_{xx} = \sigma_{xy} = \sigma_{yy} = \sigma_{xz} = 0$$

$$\rightsquigarrow \quad \varepsilon_{xx} = \varepsilon_{yy} = -\frac{1}{3}\left(\frac{1}{2\mu} - \frac{1}{3k}\right)p$$

$$\varepsilon_{zz} = \frac{1}{9k}p + \frac{1}{3\mu}p$$

ε_{zz} is the relative lengthening of the rod

$$E := \frac{p}{\varepsilon_{zz}} \qquad \text{Young's modulus}$$

$$= \frac{9k\mu}{3k+\mu}$$

$\varepsilon_{xx}, \varepsilon_{yy}$ is the relative compression in the transverse direction

$$\nu := -\frac{u_{xx}}{u_{zz}} = \frac{1}{2}\frac{(3k-2\mu)}{(3k+\mu)}$$

Poisson's ratio

$$-1 \leq \nu \leq 1/2$$

$\nu = 1/2 \longleftrightarrow$ incompressible material $(k \to \infty)$

- μ, k can be substituted by ν, E

$$\boxed{\sigma_{ik} = \frac{E}{(1+\nu)}\left(\varepsilon_{ik} + \frac{\nu}{(1-2\nu)}\varepsilon_{ll}\,\delta_{ik}\right)}$$

$$\sigma_{ll} = \frac{E}{1-2\nu}\varepsilon_{ll}$$

case $\nu = 1/2$ (pure shear), then

$$\varepsilon_{ik} = \frac{(1+\nu)}{E}\sigma_{ik}$$

I 7. Equations of equilibrium

$$\underbrace{\underline{F}_{int}}_{} + \underline{F}_{ext} = 0$$

$$\partial_j \sigma_{ij} + F_{i,ext} = 0$$

(for instance, $\underline{F}_{ext}$ can be given by gravity

$$\partial_j \sigma_{ij} + \rho g_i = 0 \quad , \quad \rho \ldots \text{ mass density}$$

$$\underline{g} = g \cdot \underline{e}_z$$
)

Substituting $\underline{u}$ into this equation, gives

$$\frac{E}{2(1+\nu)}\left(\Delta \underline{u} + \frac{1}{(1-2\nu)} \underbrace{\underline{\nabla}\, \underline{\nabla} \cdot \underline{u}}_{\text{grad div}}\right) = -\underline{F}_{ext}$$

$$\left(= -\rho\, \underline{g}\right)$$

often external forces act only on boundary, then

$$(1-2\nu)\Delta \underline{u} + \underline{\nabla}\, \underline{\nabla} \cdot \underline{u} = 0 \qquad (*)$$

it follows: $\underline{\nabla}(\,*\,) = 0$

$\Leftrightarrow \Delta \operatorname{div} \underline{u} = 0$

$$\Delta(\,*\,) = 0 \longrightarrow \Delta\Delta \underline{u} = 0$$

$\Longrightarrow$ div $\underline{u}$ is a harmonic function

$\underline{u}$ is a biharmoic vector

→ extra tools for solving equations of equilibrium

I.8 Equilibrium condition in curvilinear coordinates

example: cylindrical coordinates

$$\underline{r} = \begin{pmatrix} \rho \cos\varphi \\ \rho \sin\varphi \\ z \end{pmatrix}, \quad \text{coord.: } \varphi, \rho, z$$

basis vectors of tangent space

$$= \underline{e}_\varphi = \begin{pmatrix} -\rho\sin\varphi \\ \rho\cos\varphi \\ 0 \end{pmatrix}, \quad \partial_\rho \underline{r} = \underline{e}_\rho = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, \quad \partial_z \underline{r} = \underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

challenge: $\underline{e}_\varphi, \underline{e}_\rho$ depend on coordinates and are not constant

equilibrium condition in cartesian coord:

$$\partial_j \sigma_{ij} = 0$$

transforms to $\nabla_j \sigma_{ij} = 0$, ∇_j ... covariant derivative of the coord. system

covariant derivatives take into account variation of basis vectors of the tangent space

- rank 2 tensor $\underline{\underline{t}} = t^{jk}\, \underline{e}_j \otimes \underline{e}_k$

Application of cov. derivative

$$\nabla_i \left(t^{jk}\, \underline{e}_j \otimes \underline{e}_k\right) = \partial_i t^{jk}\, \underline{e}_j \otimes \underline{e}_k + t^{jk} \nabla_i \underline{e}_j \otimes \underline{e}_k + t^{jk}\, \underline{e}_j \otimes \nabla_i \underline{e}_k$$

$$= \partial_i t^{jk}\, \underline{e}_j \otimes \underline{e}_k + t^{jk}\, \Gamma^l_{ij}\, \underline{e}_l \otimes \underline{e}_k$$

$$+ t^{jk}\, \underline{e}_j \otimes \Gamma^l_{ik}\, \underline{e}_l$$

$$= \partial_i t^{jk}\, \underline{e}_j \otimes \underline{e}_k + \underbrace{t^{lk}\, \Gamma^j_{il}\, \underline{e}_j \otimes \underline{e}_k}_{l \leftrightarrow j}$$

$$+ \underbrace{t^{jl}\, \Gamma^k_{il}\, \underline{e}_j \otimes \underline{e}_k}_{l \leftrightarrow k}$$

$$\rightarrow \nabla_i \underline{\underline{t}} = \left(\partial_i t^{jk} + t^{lk}\, \Gamma^j_{il} + t^{jl}\, \Gamma^k_{il}\right) \underline{e}_j \otimes \underline{e}_k$$

Therefore $0 = \nabla_i\, \sigma^{ji} = \partial_i\, \sigma^{ji} + \Gamma^j_{il}\, \sigma^{li}$

$$+ \Gamma^i_{il}\, \sigma^{jl}$$

... new equilibrium condition

For cylindrical and spherical coordinates

$$\Gamma^l_{ij} = (\partial_i \underline{e}_j) \circ \underline{e}_l$$

Cylindr. coord.:

$$\underline{e}_\varphi = \begin{pmatrix} -\rho \sin\varphi \\ \rho \cos\varphi \\ 0 \end{pmatrix} \longrightarrow \partial_\varphi \underline{e}_\varphi = \begin{pmatrix} -\rho\cos\varphi \\ -\rho\sin\varphi \\ 0 \end{pmatrix} = -\rho\, \underline{e}_\rho$$

$$\rightsquigarrow \Gamma^\rho_{\varphi\varphi} = -\rho$$

$$\partial_\rho \underline{e}_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{\underline{e}_\varphi}{\rho}$$

$$\rightsquigarrow \Gamma^\varphi_{\rho\varphi} = \frac{1}{\rho}$$

$$\partial_\varphi \underline{e}_\rho = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix} = \frac{\underline{e}_\varphi}{\rho} \rightsquigarrow \Gamma^\varphi_{\varphi\rho} = \frac{1}{\rho}$$

all other components are zero

$$\nabla_i \sigma^{\varphi i} = \partial_i \sigma^{\varphi i} + \Gamma_{il}^{\varphi} \sigma^{li} + \Gamma_{il}^{i} \sigma^{\varphi l}$$

$$= \partial_i \sigma^{\varphi i} + \Gamma_{\varphi\rho}^{\varphi} \sigma^{\varphi\rho} + \Gamma_{\varphi\rho}^{\varphi} \sigma^{\varphi\rho} + \Gamma_{\varphi\rho}^{\varphi} \sigma^{\varphi\rho}$$

$$= \partial_i \sigma^{\varphi i} + \frac{3}{\rho} \sigma^{\varphi\rho}$$

now $\underline{\underline{\sigma}} = \sigma^{ij} \underline{e}_i \otimes \underline{e}_j$, $\underline{e}_\varphi$ is not normalized

$$\underline{e}_\varphi = \rho \cdot \hat{\underline{e}}_\varphi$$

$$\leadsto \quad \sigma_{\varphi\varphi} \underline{e}_\varphi \otimes \underline{e}_\varphi = \rho^2 \sigma_{\varphi\varphi} \hat{\underline{e}}_\varphi \otimes \hat{\underline{e}}_\varphi$$

$$= \hat{\sigma}_{\varphi\varphi} \hat{\underline{e}}_\varphi \otimes \hat{\underline{e}}_\varphi$$

$$\sigma_{\varphi\varphi} = \frac{\hat{\sigma}_{\varphi\varphi}}{\rho^2}, \quad \text{analogously } \sigma_{\varphi\rho} = \frac{\hat{\sigma}_{\varphi\rho}}{\rho}$$

$$\sigma_{\varphi z} = \frac{\hat{\sigma}_{\varphi z}}{\rho}$$

$$\nabla_i \sigma^{\varphi i} = \partial_\varphi \sigma^{\varphi\varphi} + \partial_\rho \sigma^{\varphi\rho} + \partial_z \sigma^{\varphi z} + \frac{3}{\rho} \sigma^{\varphi\rho} = 0$$

$$0 = \partial_\varphi \left(\frac{\hat{\sigma}_{\varphi\varphi}}{\rho^2} \right) + \partial_\rho \left(\frac{\hat{\sigma}^{\varphi\rho}}{\rho} \right) + \partial_z \frac{\hat{\sigma}^{\varphi z}}{\rho} + \frac{3}{\rho^2} \hat{\sigma}^{\varphi\rho} \quad | \cdot \rho$$

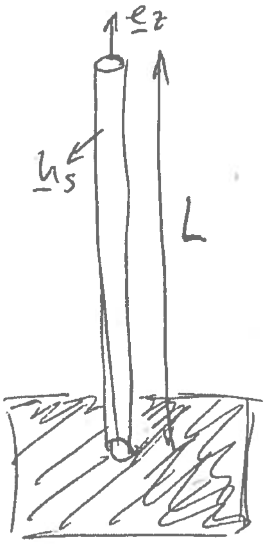
$$0 = \frac{\partial_\varphi}{\rho} (\hat{\sigma}_{\varphi\varphi}) + \partial_\rho \hat{\sigma}^{\varphi\rho} - \frac{\hat{\sigma}^{\varphi\rho}}{\rho} + \partial_z \hat{\sigma}^{\varphi z} + \frac{3}{\rho} \hat{\sigma}^{\varphi\rho}$$

$$\underbrace{0 = \frac{\partial_\varphi}{\rho} \hat{\sigma}_{\varphi\varphi} + \partial_\rho \hat{\sigma}^{\varphi\rho} + \partial_z \hat{\sigma}^{\varphi z} + \frac{2}{\rho} \hat{\sigma}^{\varphi\rho}}_{\text{text book equation}}$$

$\nabla_i \sigma^{\rho i}$, $\nabla_i \sigma^{zi}$ follow accordingly

I. 9. Simple examples

1) Determine the deformation of a long rod standing vertically due to gravity



$$\begin{pmatrix} \partial_i \sigma_{xi} \\ \partial_i \sigma_{yi} \\ \partial_i \sigma_{zi} \end{pmatrix} = \varrho \cdot g \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

mass density

no forces at the boundary at the top and on the sides

top: $\underline{\underline{\sigma}} \cdot \underline{e}_z = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = \left.\begin{pmatrix} \sigma_{xz} \\ \sigma_{yz} \\ \sigma_{zz} \end{pmatrix}\right|_{z=L}$

sides : $\underline{\underline{\sigma}} \cdot \underline{n}_s = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longrightarrow \sigma_{ij} = 0$ except σ_{zz}

assume no x-y-dependence of σ_{ij}

Equilibrium cond. reads

$$\partial_z \sigma_{zz} = \varrho \cdot g$$

$$\sigma_{zz} = \varrho \cdot g \cdot z + a(x,y)$$

bound. cond. $\sigma_{zz}(L) = 0 = \varrho \cdot g \cdot L + a(x,y)$

$\rightsquigarrow a = -\varrho g L$

$\sigma_{zz} = \varrho \cdot g (z-L)$

$\varepsilon_{ij} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu \sigma_{ll} \delta_{ij}\right]$

$\rightarrow \varepsilon_{zz} = \frac{1}{E} \varrho g (z-L)$, $\varepsilon_{xx} = \varepsilon_{yy} = -\frac{\nu}{E} \varrho g (z-L)$

$\varepsilon_{xy} = \varepsilon_{zy} = \varepsilon_{xz} = 0$

By integration, we get displacements

$$\varepsilon_{xx} = \partial_x u_x \longrightarrow u_x = -\frac{\nu \rho g (z-L) x}{E}$$

$$u_y = -\frac{\nu \rho g (z-L) y}{E}$$

$$u_z = +\frac{\rho g}{2E}\left((L-z)^2 - L^2\right) + f_z(x,y)$$

$$\partial_x u_z + \partial_z u_x \overset{!}{=} 0$$

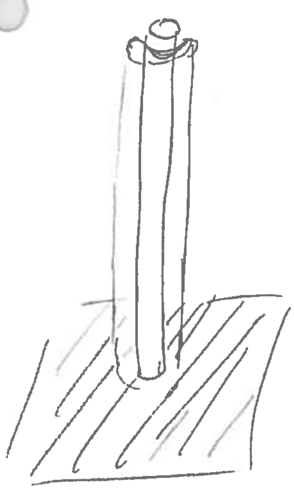
$$\partial_y u_z + \partial_z u_y \overset{!}{=} 0$$

$$\left.\begin{array}{l} \partial_x f_z = +\frac{\nu \rho g}{E} \cdot x \\ \partial_y f_z = +\frac{\nu \rho g}{E} \cdot y \end{array}\right\} f_z = +\frac{\nu \rho g}{2E}(x^2+y^2)$$

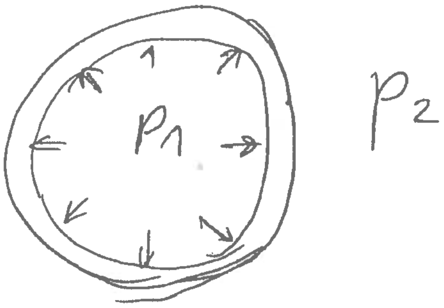
$$\rightarrow u_z = +\frac{\rho g}{2E}\left[\left((L-z)^2 - L^2\right) + \nu(x^2+y^2)\right]$$

$u_z(0) = 0$ only holds at $z=0$

$\hookrightarrow$ inconsistency $\rightsquigarrow$ solution does not hold at bottom of the rod



2) Determine the deformation of a hollow sphere with pressure p_1 inside, p_2 outside



rotational symmetry
$\rightarrow \underline{u} = u_r(r) \cdot \underline{e}_r$
(no φ, or θ dependence, no displacement comp. parallel to $\underline{e}_\varphi$ or $\underline{e}_\theta$)

$$\rightsquigarrow \quad \varepsilon_{rr} = \partial_r u_r$$

$$\varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta} = \frac{u_r}{r}, \quad \varepsilon_{r\theta} = \varepsilon_{r\varphi} = \varepsilon_{\theta\varphi} = 0$$

$$(1-2\nu)\underbrace{\Delta \underline{u}}_{} + \text{grad div } \underline{u} = 0$$ $\leftarrow$ no external forces, apart from boundaries

$$(1-2\nu)\left(-\underline{\nabla} \times \underline{\nabla} \times \underline{u} + \text{grad div } \underline{u}\right) + \text{grad div } \underline{u} = 0$$

$$\underline{\nabla} \times \underline{u} = 0 \quad \longrightarrow \quad \text{grad div } \underline{u} = 0$$

$$\text{grad}\left(\frac{1}{r^2}\partial_r r^2 u_r\right) = 0$$

$$\partial_r \frac{1}{r^2}\partial_r\left(r^2 u_r\right) = 0$$

$$\frac{1}{r^2}\partial_r\left(r^2 u_r\right) = \text{const.} =: 3a$$

$$\partial_r(r^2 u_r) = 3ar^2$$

$$u_r = ar + \frac{b}{r^2}$$

$$\varepsilon_{rr} = a - \frac{2b}{r^3} \qquad \varepsilon_{ll} = \operatorname{tr} \varepsilon = 3a$$

$$\varepsilon_{\theta\theta} = a + \frac{b}{r^3} \qquad \sigma_{rr} = \frac{E}{(1+\nu)}\left(\varepsilon_{rr} + \frac{\nu}{1-2\nu}\varepsilon_{ll}\right)$$

$$\varepsilon_{\varphi\varphi} = \varepsilon_{\theta\theta}$$

$$\sigma_{rr} = \frac{E}{(1+\nu)}\left(a - \frac{2b}{r^3} + \frac{\nu}{1-2\nu}a\right) = \frac{E}{1-2\nu}a - \frac{2Eb}{(1+\nu)r^3}$$

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{E}{(1+\nu)}\left(a + \frac{b}{r^3} + \frac{\nu}{1-2\nu}a\right) = \frac{E}{1-2\nu}a + \frac{Eb}{(1+\nu)r^3}$$

boundary conditions

$$\sigma_{rr}(R_2) = -p_2 \quad , \quad \sigma_{rr}(R_1) = -p_1$$

for simplicity, sét $p_2 = 0$

$$\frac{Ea}{(1-2\nu)} = \frac{2bE}{(1+\nu)R_1^3} - p_1$$

$$\frac{Ea}{(1-2\nu)} = \frac{2bE}{(1+\nu)R_2^3} - p_2$$

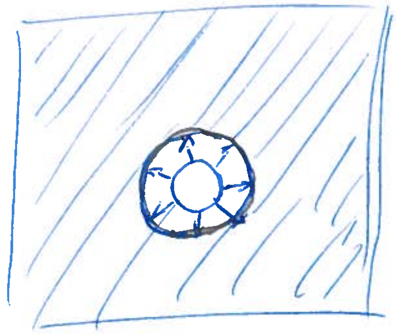
$$b = \frac{(1+\nu)}{2E} R_1^3 R_2^3 \frac{(p_1 - p_2)}{R_2^3 - R_1^3}$$

$$a = \frac{(1-2\nu)}{E} \frac{p_1 R_1^3 - p_2 R_2^3}{R_2^3 - R_1^3}$$

3) Example: Growing tumor spheroid in a gel

wanted: elastic stress building up in the gel due to the expanding tumor

given: original tumor radius R_0, new radius R_1
Young's modulus E of the gel
Poisson ratio: ν



due to symmetry reasons

$$\underline{u}(\underline{r}) = u_r(r)\,\underline{e}_r$$

Analogous to the situation of a hollow sphere, we find

$$\operatorname{div}\underline{u} = \frac{1}{r^2}\partial_r(r^2 u_r) = \text{const.}$$

$$\rightsquigarrow u_r(r) = ar + b/r^2$$

As we need $u_r(\infty) = 0 \to a = 0$

$$u_r = \frac{b}{r^2}$$

$$\varepsilon_{rr} = \partial_r u_r = -\frac{2b}{r^3}, \quad \varepsilon_{\theta\theta} = \varepsilon_{\varphi\varphi} = \frac{u_r}{r} = \frac{b}{r^3}$$

boundary condition $u_r(R_0) = R_1 - R_0 =: \Delta R$

$$\to b = R_0^2 \Delta R$$

$$\operatorname{tr}\varepsilon = -\frac{2b}{r^3} + 2\times\frac{b}{r^3} = 0 \longrightarrow \text{no volume changes}$$

$$\sigma_{rr} = \frac{E}{(1+\nu)}\left(\varepsilon_{rr} + \frac{\nu}{(1-2\nu)}\underbrace{\operatorname{tr}\varepsilon}_{0}\right)$$

$$= \frac{-E}{(1+\nu)}\frac{2b}{r^3} = \frac{-2E}{(1+\nu)}\frac{R_0^2\Delta R}{r^3}$$

$$\sigma_{rr}(R_0) = -\frac{2E}{(1+\nu)} \frac{\Delta R}{R_0}$$

Pressure felt at the surface of the tumor spheroid $\frac{2E}{(1+\nu)} \frac{\Delta R}{R_0}$

$\rightarrow$ hinders further expansion of tumor

$$\sigma_{\theta\theta} = \sigma_{\varphi\varphi} = \frac{E}{(1+\nu)} \left(\varepsilon_{\theta\theta/\varphi\varphi} + \frac{\nu}{(1-2\nu)} \underbrace{tr\,\varepsilon}_{0} \right)$$

$$= \frac{E}{(1+\nu)} \frac{b}{r^3} = \frac{E}{(1+\nu)} \frac{\Delta R R_0^2}{r^3}$$

$$\sigma_{\varphi\varphi}(R_0) = \sigma_{\theta\theta}(R_0) = \frac{E}{(1+\nu)} \frac{\Delta R}{R_0}$$

... like a surface tension for the spheroid! $\rightarrow$ enforces round shape

consider the case $R_1 = 2R_0 \rightarrow \Delta R = R_0$

$$\sigma_{rr}(R_0) = \frac{2E}{(1+\nu)} \frac{R_0}{R_0} = \frac{2E}{(1+\nu)}$$

$$\varepsilon_{rr}(R_0) = -\frac{2b}{R_0^3} = -\frac{2R_0^2 \Delta R}{R_0^3} = -2\frac{\Delta R}{R_0}$$

$$= -2$$

What is wrong here?

VI Pappkovich Neuber approach and Green's function

VI.1. Pappkovich Neuber ansatz

Governing equation of elasticity

$$\underbrace{\frac{E}{2(1+\nu)}\left(\frac{1}{(1-2\nu)}\underline{\nabla}\,\mathrm{div}\,\underline{u} + \Delta\underline{u}\right)}_{\partial_i \sigma_{ij}} + \underbrace{\underline{F}_{ext}}_{\substack{\text{volumic}\\ \text{force}}} = 0$$

such as gravity

We discuss simpler case where $\underline{F}_{ext} = 0$

$$\underbrace{\frac{1}{(1-2\nu)}\underline{\nabla}\,\mathrm{div}\,\underline{u} + \Delta\underline{u} = 0}_{=:(*)}$$

$$\underline{\nabla}\circ(*) = \frac{1}{(1-2\nu)}\,\mathrm{div}\,\Delta\underline{u} + \Delta\,\mathrm{div}\,\underline{u} = 0$$

$$\rightarrow \Delta\,\mathrm{div}\,\underline{u} = 0 \rightarrow \mathrm{div}\,\underline{u} \text{ is harmonic}$$

$$\Delta(*) = \frac{1}{1-2\nu}\underline{\nabla}\underbrace{\Delta\,\mathrm{div}\,\underline{u}}_{0} + \Delta^2 u = 0$$

$$= \Delta^2 u = 0 \rightarrow \underline{u} \text{ is biharmonic}$$

Theorem

- Generally $\underline{u}$ can be written in the form

$$\boxed{\underline{u} = 4(1-\nu)\underline{B} - \underline{\nabla}(\underline{r}\cdot\underline{B} + B_0) + \underline{\nabla}\chi_0}$$

Pappkovich-Neuber ansatz

where $\underline{B}$ is a harmonic vector
B_0 is a harmonic function
χ_0 is a particular solution to

the equation $\Delta \chi_0 = \frac{1-2\nu}{2G(1-\nu)} \phi$,

where $\underline{F}_{ext} = -\underline{\nabla} \phi$

<u>Proof for the case $F_{ext} = 0$:</u>

$$\Delta(\underline{r} \cdot \operatorname{div} \underline{u}) = 2 \underline{\nabla} \operatorname{div} \underline{u}$$

$$\left(\sum_j \partial_j^2 (r_i \operatorname{div} \underline{u}) = \sum_j \Big[(\partial_j^2 \operatorname{div} \underline{u}) r_i + (\partial_j^2 r_i) \operatorname{div} \underline{u} + 2 \partial_j r_i \, \partial_j \operatorname{div} \underline{u} \Big] \right)$$

$$= \overbrace{\Delta \operatorname{div} \underline{u}}^{0} \, r_i + \overbrace{\Delta r_i}^{0} \operatorname{div} \underline{u} + 2 \partial_i \operatorname{div} \underline{u}$$

We replace $\underline{\nabla} \operatorname{div} \underline{u}$ in (*) by $\frac{\Delta(\underline{r} \operatorname{div} \underline{u})}{2}$ and get

$$\Delta \Big(\underbrace{\frac{\operatorname{div} \underline{u}}{2(1-2\nu)} \underline{r} + \underline{u}}_{\text{harmonic vector}} \Big) = 0$$

We make the ansatz

$$\underline{u} = 4(1-\nu) \underline{B} + \underline{\nabla} \chi,$$

$$\operatorname{div} \underline{u} = 4(1-\nu) \operatorname{div} \underline{B} + \Delta \chi$$

Force balance equation (*) reads with this ansatz

$$0 = \frac{1}{(1-2\nu)} \underline{\nabla} \big(4(1-\nu) \operatorname{div} \underline{B} + \Delta \chi \big) + \Delta \big(\underbrace{4(1-\nu) \underline{B}}_{0} + \underline{\nabla} \chi \big)$$

$$= \frac{1}{(1-2\nu)} \underline{\nabla} \big[4(1-\nu) \operatorname{div} \underline{B} + (2-2\nu) \Delta \chi \big]$$

$$\longrightarrow \Delta \chi = -2 \operatorname{div} \underline{B}$$

$$\chi = \chi_{part} + \chi_{hom}$$

$\chi_{part} = -\underline{r} \cdot \underline{B}$ particular solution

$\chi_{hom} =: -B_0$ arbitrary harmonic function

$\square$

Within the Pappkovich Neuber approach ($F_{ext} = 0$) the stress tensor takes the form

$$\sigma_{ij} = 2G \Big[2\nu\, \delta_{ij} \operatorname{div} \underline{B} + (1-2\nu)(\partial_i B_j + \partial_j B_i) - r_k \partial_i \partial_j B_k - \partial_i \partial_j B_0 \Big]$$

Why is the Pappkovich Neuber ansatz useful? There is a big mathematical toolbox for harmonic functions/vectors that we can exploit

II.2. Green's function for unbounded elastic medium – Thomson's solution

problem: elastic unbounded medium and a distribution of forces $\underline{f}(\underline{r})$ acting inside. We need to solve the force balance equation

$$\frac{E}{2(1+\nu)} \left(\frac{1}{1-2\nu} \nabla \operatorname{div} \underline{u} + \Delta \underline{u} \right) =: L \underline{u} = - \underline{f}(\underline{r})$$

solution strategy: find fundamental solution $\underline{\underline{G}}(\underline{r})$ (Green's function)

such that

$$L\left(\underline{\underline{G}}(\underline{r}) \cdot \underline{v} \right) = \underline{v}\, \delta(\underline{r})$$

or

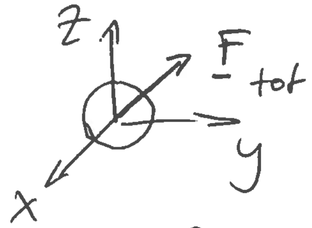
$$L\, \underline{\underline{G}}(\underline{r}) = \delta(\underline{r})\, \mathbb{1}$$

then $\underline{u}(\underline{r}) = \int_V G(\underline{r}-\underline{r}') f(\underline{r}')\, dr'$

$$L\underline{u} = \int_V L G(\underline{r}-\underline{r}') f(\underline{r}')\, dr'$$

$$= \int_V \delta(\underline{r}-\underline{r}') f(\underline{r}')\, dr' = f(\underline{r})$$

To calculate $\underline{\underline{G}}(\underline{r})$ we consider the scenario of a point force at the origin $\underline{f}(\underline{r}) = \delta(\underline{r})\, \underline{F}_{tot}$



We have for any sphere around the origin

$$-\underline{F}_{tot} = \int_{Sphere} -\delta(\underline{r})\, \underline{F}_{tot}\, dV = \int_{Sphere} \mathrm{div}\, \underline{\underline{\sigma}} \cdot dV = \int_{\partial Sphere} \underline{\underline{\sigma}} \cdot d\underline{A}$$

$$= \int_{\partial Sphere} \underline{\underline{\sigma}} \cdot \underline{n}\, dA$$

$$= R^2 \int_{\partial(\text{unit sphere})} \underline{\underline{\sigma}} \cdot \underline{n}\, dA$$

$$\rightarrow \sigma \propto \frac{1}{R^2} \rightarrow \underline{u} \propto \frac{1}{R}$$

Pappkovich Neuber ansatz with $\underline{B} = \frac{A}{r} \underline{F}_{tot}$

$B_0 = 0$

$$\underline{u} = 4(1-\nu)\, \underline{B} - \underline{\nabla}(\underline{r} \cdot \underline{B} + \underline{B}_0)$$

$$= A\left[(3-4\nu) \frac{\underline{F}_{tot}}{r} + \frac{\underline{F}_{tot} \cdot \underline{r}}{r^3}\, \underline{r} \right] \quad (*)$$

using $\varepsilon_{ij} = \frac{(\partial_i u_j + \partial_j u_i)}{2}$

$\sigma_{ij} = \frac{E}{(1+\nu)}\left(\varepsilon_{ij} + \frac{\nu}{(1-2\nu)}\varepsilon_{ll}\,\delta_{ij}\right)$

$$\sigma_{ij} = \frac{2GA}{r^3}\Big[(1-2\nu)(\underline{F}_{tot}\cdot\underline{r})\,\delta_{ij} - F_i^{tot} r_j - F_j^{tot} r_i - 3\,\frac{\underline{F}_{tot}\cdot\underline{r}}{r^2}\,r_i r_j\Big]$$

Calculating now

$$\int_{\text{sphere}} \underline{\underline{\sigma}}\cdot\underline{n}\;dA \overset{!}{=} -\underline{F}_{tot}$$

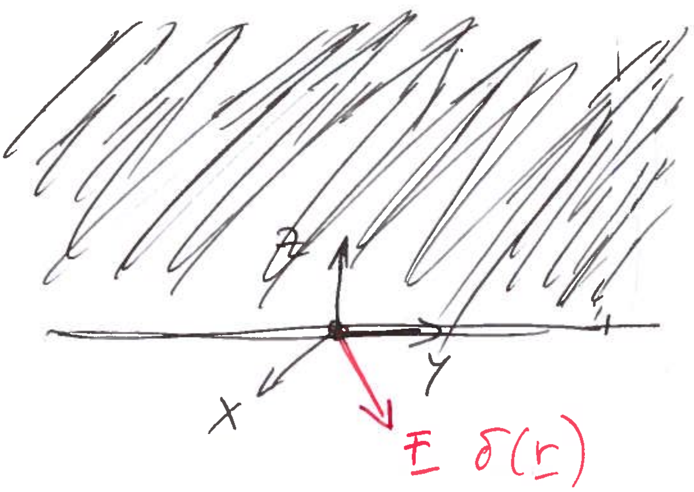
determines $A = \frac{1}{16\pi G(1-\nu)}$

Comparing with (*) gives

$$\hat{\underline{\underline{G}}}(\underline{r}) = \frac{1}{16\pi G(1-\nu)r}\Big[(3-4\nu)\,\mathbb{1} + \frac{\underline{r}\,\underline{r}}{r^2}\Big]$$

$\equiv$ Thomson's solution

II.3 Green's function for the elastic half space – Boussinesq-Cerruti solution

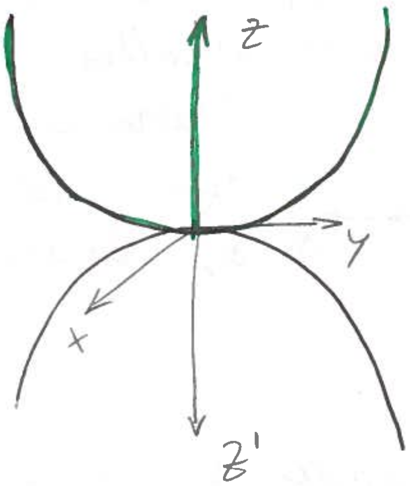


- solution for point force acting on surface of a sphere (at the origin)
- uses Pappkovich-Neuber approach (Landau § 8)

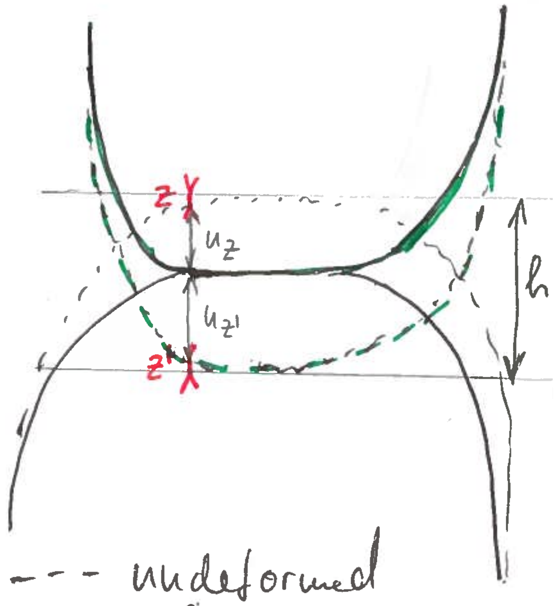
$$\underline{u} = \frac{(1+\nu)}{2\pi E r}\begin{pmatrix} -\frac{(1-2\nu)}{r} x F_z + 2(1-\nu) F_x + \frac{2\nu x}{r^2}(x F_x + y F_y) \\ -\frac{(1-2\nu)}{r} y F_z + 2(1-\nu) F_y + \frac{2\nu y}{r^2}(\text{— " —}) \\ 2(1-\nu) F_z + \frac{(1-2\nu)}{r}(x F_x + y F_y) \end{pmatrix}$$

III. Contact mechanics

- imagine the following problem:
we have two bodies (e.g. two spheres) that make contact at one point initially and then are forced more together by a force $\underline{F}$
$\Rightarrow$ the area of contact increases



force switched on



Questions:
How large is the approach and how large is the contact area in dependence of the acting force

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We find that $(z+u_z)+(z'+u_z')=h$

$z = \kappa_{\alpha\beta}\, x_\alpha x_\beta\,,\quad \alpha,\beta = 1,2$

$z' = \kappa'_{\alpha\beta}\, x_\alpha x_\beta$

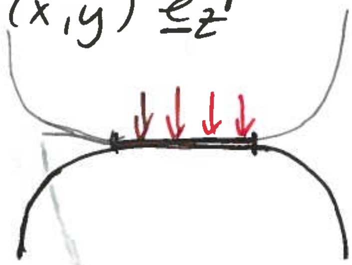
$(\kappa_{\alpha\beta}+\kappa'_{\alpha\beta})\, x_\alpha x_\beta + u_z + u_z' = h$

Choose coord. system in x-y-plane such that $(\kappa_{\alpha\beta} + \kappa'_{\alpha\beta})$ is symmetric

$$= \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}$$

$$\rightarrow (Ax^2 + By^2) + u_z + u'_z = 0 \qquad (*)$$

- force distribution along the contact area $f(x,y)\, \underline{e}_{z'}$



* assume that contact area and approach distance h are much smaller then curvature radii of bodies

top view contact area

→ it is elliptical

Due to small curvature of body surface, it is well approximated by an elastic half space. We have found a Green's function for point force acting on surface of an elastic half space (Boussinesq solution).
Surface forces point only along the z (z') direction.

Boussinesq solution: $u_z^{\text{Bouss.}} = \frac{(1-\nu^2)}{\pi E} \frac{1}{r} F_z$

For a distribution $f(x,y)\, \underline{e}_z$, we have

$$u_z = \frac{(1-\nu^2)}{\pi E} \int \frac{f(x',y')}{\underbrace{\sqrt{[(x-x')^2 + (y-y')^2 + z^2]}}_{=:r}} dx'dy'$$

$$\rightsquigarrow \frac{u_z}{u_z'} = \text{const.} = \frac{(1-\nu^2)E'}{(1-\nu'^2)E}$$

Inserting this ansatz for u_z ($u_{z'}$) into (*) gives

$$\frac{1}{\pi}\left(\frac{1-\nu^2}{E} + \frac{1-\nu'^2}{E'}\right)\int \frac{f(x',y')}{r(x',y')}\,dx'\,dy' = h - Ax^2 - By^2 \quad (*)$$

$\equiv$ differential equation in $f(x,y)$ that needs to be solved

Ansatz: $f(x,y) = \text{const.}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}$

Constraint: $\int_{\text{contact area}} f(x,y)\,dA = |F_{tot}|$ gives $\text{const.} = \frac{3F_{tot}}{2\pi ab}$

One can show that

$$\int \frac{\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}{r(x',y')}\,dx'\,dy' = \frac{1}{2}\pi ab\int_0^\infty\left(1-\frac{x^2}{a^2+\xi}-\frac{y^2}{b^2+\xi}\right)\frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}}$$

Substituting into (*) gives

$$\frac{DF_{tot}}{\pi}\int_0^\infty\left(1-\frac{x^2}{(a^2+\xi)}-\frac{y^2}{(b^2+\xi)}\right)\frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}} = h - Ax^2 - By^2$$

for all x and y

$$D = \frac{3}{4}\left(\frac{1-\nu^2}{E} + \frac{1-\nu'^2}{E'}\right)$$

Therefore, we find

$$h = \frac{F_{tot} D}{\pi} \int_0^\infty \frac{d\xi}{\sqrt{(a^2+\xi)(b^2+\xi)\xi}}$$

$$A = \frac{F_{tot} D}{\pi} \int_0^\infty \frac{d\xi}{(a^2+\xi)\sqrt{\cdots}}$$

$$B = \frac{F_{tot} D}{\pi} \int_0^\infty \frac{d\xi}{(b^2+\xi)\sqrt{\cdots}}$$

$\longrightarrow$ sets a and b for given body geometries (A, B) and given F_{tot}

Discuss now the case of contacting spheres with radii R_1 and R_2,

then $A = B = \frac{1}{2R_1} + \frac{1}{2R_2}$, $a = b$ $\equiv$ circular contact area

set $R := \frac{1}{1/R_1 + 1/R_2}$

$$a = F_{tot}^{1/3} (D \cdot R)^{1/3} \quad \text{contact radius}$$

$$h = F_{tot}^{2/3} \left[\frac{D^2}{R}\right]^{1/3} \Rightarrow F_{tot} \propto h^{3/2}$$

... distance of approach

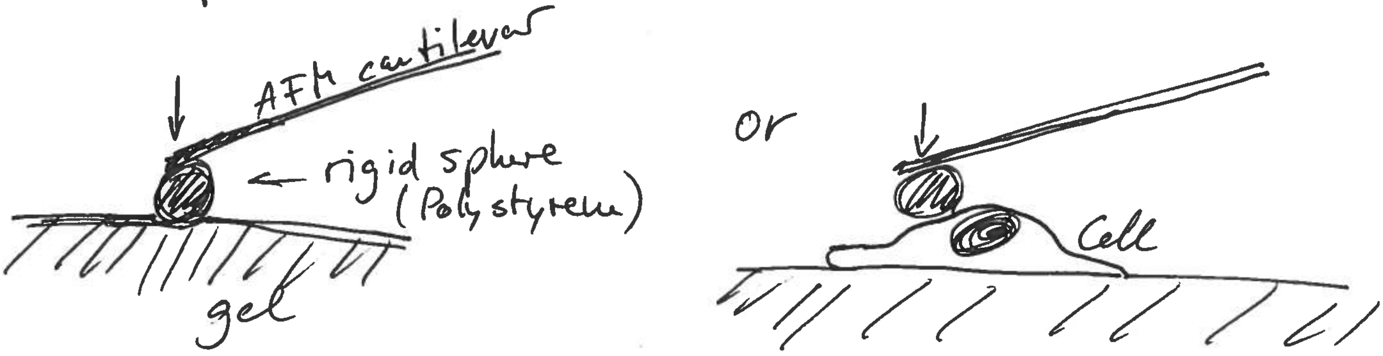
Consider $R_2 \gg R_1$, $E_2 \ll E_1$

(small rigid sphere$^{(1)}$ indenting a large soft sphere$^{(2)}$)

$$R \approx R_1, \quad D \cong \frac{3}{4}\left(\frac{1-\nu_2^2}{E_2}\right)$$

$$h = F_{tot}^{2/3}\left[\frac{9}{16}\frac{(1-\nu_2^2)^2}{E_2^2 R_1}\right]^{1/3} \quad (*)$$

→ used for stiffness measurements in terms of AFM on soft substrates



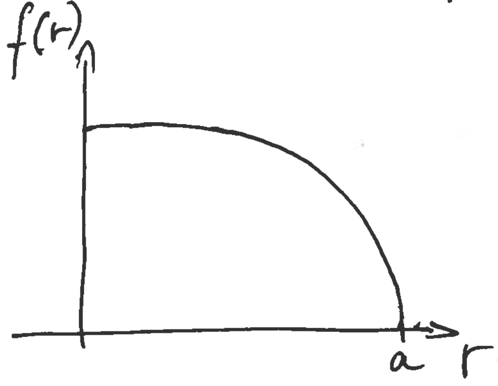
AFM output: force F, indentation depth h

→ get $F(h)$

→ fit the above dependence $(*)$ to get $\frac{E_2}{(1-\nu_2^2)}$

for contacting spheres, the pressure distribution over the contact area is

$$f(r) = \frac{3F}{2\pi a^2}\sqrt{1-\frac{r^2}{a^2}}$$



peak pressure $= 1.5 \cdot \bar{p}$

Viscoelasticity - beyond solid and liquid

- Consider a shear stress

<u>liquid:</u> $\sigma_{shear} = \eta(\dot{\varepsilon})\,\dot{\varepsilon}$

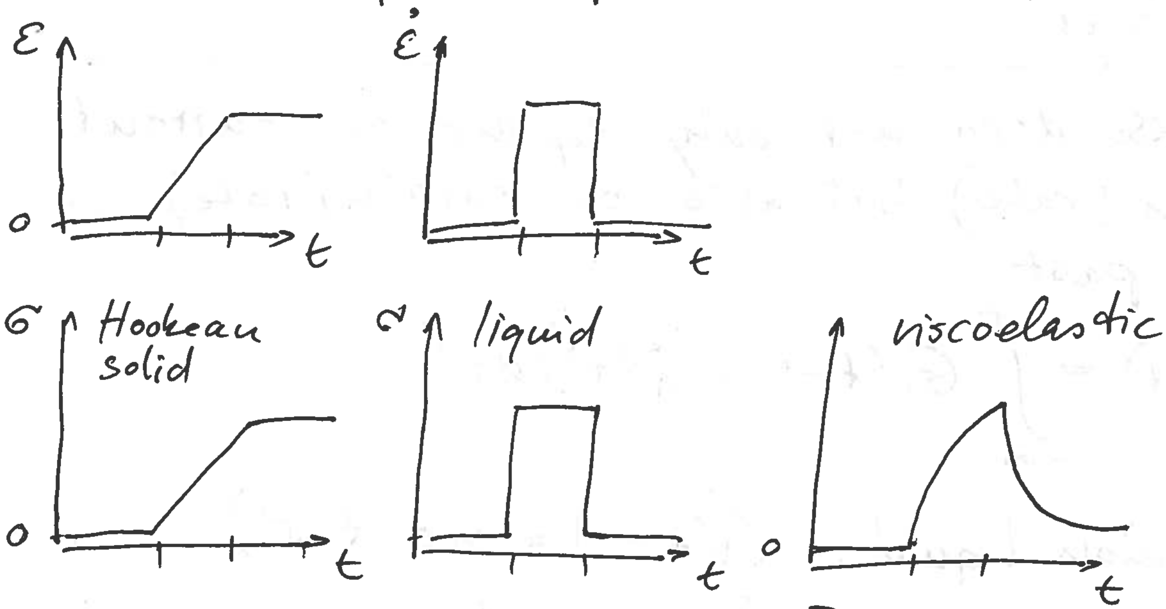
Newtonia liquid $\sigma_{shear} = \eta \cdot \dot{\varepsilon}$

<u>solid:</u> $\sigma_{shear} = F(\varepsilon)$

Hookean solid $\sigma_{shear} = E\,\varepsilon$

<u>viscoelastic material</u> $\sigma = F(\varepsilon, \dot{\varepsilon})$

- Behaviour upon step strain (shear)



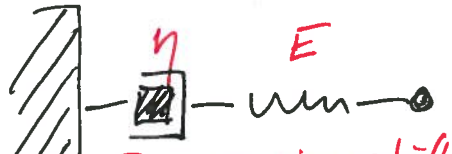
Examples:
polymer melt,
biological material

Maxwell suggested in 1867 the following dynamics for shear stress and strain in visoelastic material

$$\frac{d\varepsilon(t)}{dt} = \underbrace{\frac{1}{E}\frac{d\sigma}{dt}}_{\substack{\text{elastic}\\ \dot{\varepsilon} = \dot{\sigma}/E}} + \underbrace{\frac{\sigma}{\eta}}_{\substack{\text{liquid}\\ \dot{\varepsilon} = \sigma/\eta}} \Rightarrow \dot{\sigma} = E\dot{\varepsilon} - \frac{\sigma}{\tau}$$

$$\tau := \frac{\eta}{E}$$

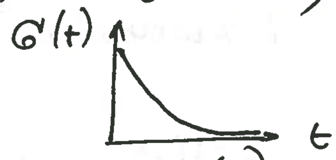
mechanical analogue (Maxwell element)



E ... spring stiffness
η ... dashpot friction

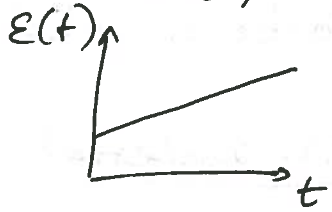
- spring in series with a dashpot

- behaviour upon step strain $\varepsilon(t) = \varepsilon_0 \Theta(t)$, then $\sigma(t) = E \cdot \varepsilon_0 \exp(-t/\tau)$



- behavior upon step stress $\sigma(t) = \sigma_0 \cdot \Theta(t)$

$$\varepsilon(t) = \frac{\sigma_0}{\eta} \cdot t + \frac{\sigma_0}{E}$$



On short time scales elastic, on long time scales fluid-like

More general desription of viscoelastic behaviour

- stress does not only depend on current strain (rate) but also on strain (rate) in the past

$$\sigma(t) = \int_{-\infty}^{t} G(t-t') \, \dot{\varepsilon}(t') \, dt'$$

Newtonian liquid: $G(t-t') = \eta \, \delta(t-t')$

$$\longrightarrow \sigma(t) = \int_{-\infty}^{t} \eta \, \delta(t-t') \, \dot{\varepsilon}(t') \, dt'$$

$$= \eta \, \dot{\varepsilon}(t)$$

Maxwell material: $G(t-t') = E \, e^{-t/\tau}$

$$\longrightarrow \sigma(t) = \int_{-\infty}^{t} E \, e^{-\frac{(t-t')}{\tau}} \, \dot{\varepsilon}(t') \, dt'$$

$$\dot{\varepsilon}(t) = \delta(t) \cdot \varepsilon_0 \quad = \quad E \, e^{-t/\tau} \quad \text{as calculated above}$$

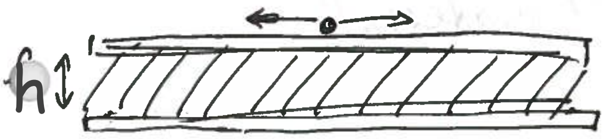
- $G(t)$ is called relaxation modulus

$$G(t) = \frac{\sigma(t)}{\varepsilon_0}, \text{ after step strain}$$

- Analogously, the notion of creep compliance $J(t)$ can be introduced as

$$J(t) = \frac{\varepsilon(t)}{\sigma_0}, \text{ after step stress}$$

- Consider an oscillatory shear (flow) in a material in between two plates



upper plate moved with speed

$$V(t) = \delta \cdot \omega \cdot \cos \omega t$$

$$\dot{\varepsilon}(t) = \frac{\delta}{h} \cdot \omega \cos \omega t = \dot{\varepsilon}_0 \cdot \cos \omega t$$

$$\varepsilon(t) = \frac{\delta}{h} \sin \omega t = \varepsilon_0 \cdot \sin \omega t$$

$$\sigma(t) = \int_{-\infty}^{t} G(t-t')\, \dot{\varepsilon}(t')\, dt'$$

$$= \int_{-\infty}^{t} G(t-t')\, \dot{\varepsilon}_0 \cdot \cos \omega t'\, dt'$$

$$= \int_{0}^{\infty} G(s)\, \dot{\varepsilon}_0 \cos(\omega(t-s))\, ds$$

$$= \int_{0}^{\infty} G(s)\, \dot{\varepsilon}_0 \left[\cos \omega t \cos \omega s + \sin \omega t \sin \omega s\right] ds$$

$$= G'(\omega)\, \varepsilon_0 \sin \omega t + G''(\omega)\, \varepsilon_0 \cos \omega t$$

where $G'(\omega) := \int_{0}^{\infty} \omega G(s) \sin \omega s\, ds =: \eta''(\omega)\, \omega$

$$G''(\omega) := \int_{0}^{\infty} \omega G(s) \cos \omega s\, ds =: \eta'(\omega) \cdot \omega$$

$G^*(\omega) = G'(\omega) + i\,G''(\omega)$ ... complex elastic modulus

$\eta^*(\omega) = \eta'(\omega) - i\,\eta''(\omega)$ ... complex viscosity

We have that if $\varepsilon(t) = \varepsilon_0 e^{i\omega t}$, then

$$\boxed{\sigma(\omega) = G^*(\omega)\cdot \varepsilon_0 e^{i\omega t}}$$

(linear relationship only valid for small strains, linear depend. up to 20% strain for polymer melt, up to only 0.1% for bread dough)

- Calculate $G^*(\omega)$ for Maxwell material

$G(t) = E e^{-t/\tau}$, then

$$G'(\omega) = \int_0^\infty \omega\, E\, e^{-t/\tau} \sin\omega t \, dt$$

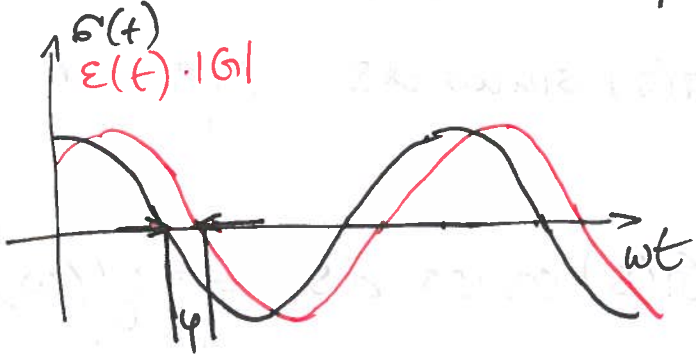
$$= \frac{\omega^2 E}{\omega^2 + 1/\tau^2}$$

$$G''(\omega) = \int_0^\infty \omega\, G(t) \cos\omega t \, dt = \frac{\omega/\tau}{\frac{1}{\tau^2} + \omega^2} E$$

$$G^*(\omega) = \frac{i\omega E}{(i\omega + 1/\tau)} \quad \left(= G'(\omega) + i\,G''(\omega)\right)$$

<u>Comment</u>: as the elastic modulus is a complex number, stress and strain for oscillatory perturbations are phase shifted

$G^*(\omega) = |G|\, e^{i\varphi(\omega)} \longrightarrow$ phase shift is $\varphi(\omega)$



$\varphi = 0$ for solids

$\Longleftrightarrow \sigma \propto \varepsilon$

$\varphi = 90°$ for liquids

$\Longleftrightarrow \sigma \propto \dot{\varepsilon}$

for $0 < \varphi(\omega) < 90°$, the material is viscoelastic at the given frequency

- for the Maxwell element we have

$$\tan(\varphi(\omega)) = \frac{1}{\tau \cdot \omega}$$

$$\longrightarrow \lim_{\omega \to \infty} \varphi(\omega) = 0 \, , \quad \lim_{\omega \to 0} \varphi(\omega) = 90°$$

$\rightsquigarrow$ on short time scales, the Maxwell material behaves like an elastic solid $\left(\frac{1}{\omega} \ll \tau\right)$

$\rightsquigarrow$ on long time scales $\left(\frac{1}{\omega} \gg \tau\right)$, the Maxwell material behaves like a liquid

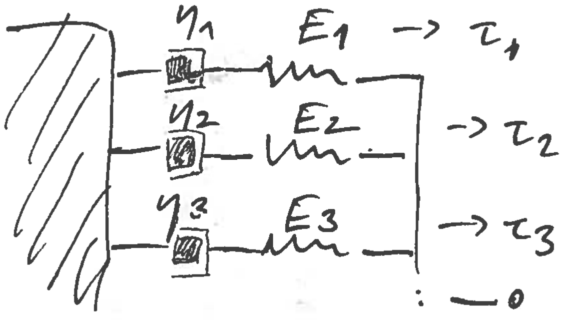
- $G'(\omega)$ ... storage modulus
  $G''(\omega)$ .. loss modulus $\left(G''(\omega) = \omega \underbrace{\eta'(\omega)}_{\text{storage viscosity}} \right)$

ratio $\frac{G''(\omega)}{G'(\omega)}$ tells about solid or liquid-like nature

$$\frac{G''(\omega)}{G'(\omega)} \gg 1 \quad \text{liquid-like}$$

$$\frac{G''(\omega)}{G'(\omega)} \ll 1 \quad \text{solid-like}$$

- <u>Generalized Maxwell model</u>
  if several time scales are needed to describe a material, one can use several Maxwell elements in parallel



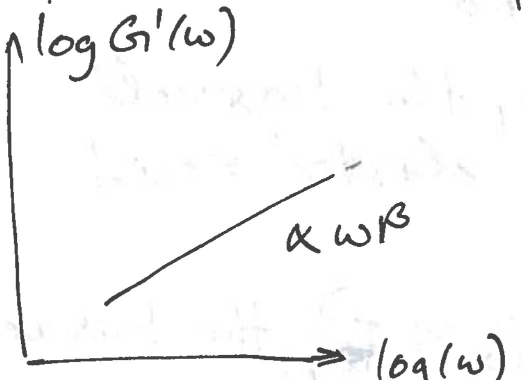
continous spectrum gives

$$G(t) = \int_0^\infty H(\tau)\, e^{-t/\tau} \frac{d\tau}{\tau}$$

$H(\tau)$ ... relaxation spectrum

- frequently complex materials show a power-law dependence of $G^*(\omega)$.
This has also been reported for cells and tissues.

If $G'(\omega) \propto \omega^\beta$, $0 < \beta < 1$



then

$$G(t) = \frac{2}{\pi} \int_0^\infty \frac{G'(\omega)}{\omega} \sin \omega t \, d\omega$$

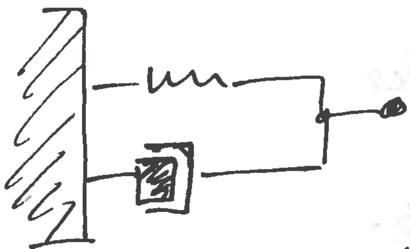
$$\propto t^{-\beta}$$

$\rightarrow$ power law decay of relaxation modulus

$\equiv$ power law decay of stress upon step strain

no characteristic time scale(s) τ here!

- Kelvin-Voigt element



$$G(t) = \eta \delta(t) + E$$

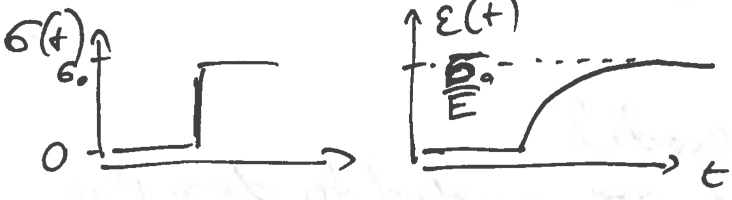
$$\sigma(t) = E \varepsilon(t) + \eta \dot{\varepsilon}(t)$$

models creep in materials

upon step stress

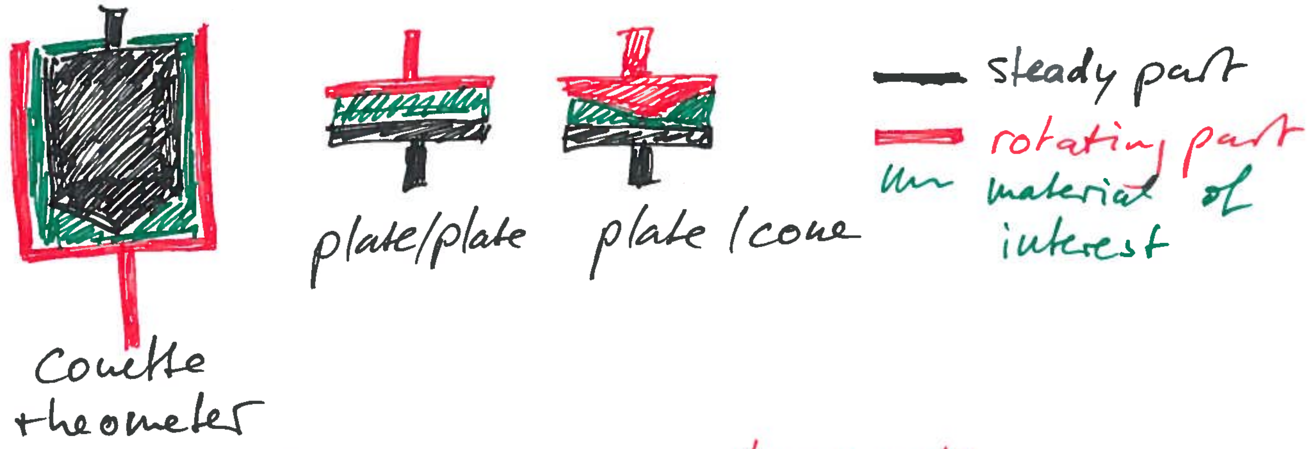
$$\sigma(t) = \sigma_0 \Theta(t)$$

$$\varepsilon(t) = \frac{\sigma_0}{E}\left(1 - e^{-t/\tau}\right)$$



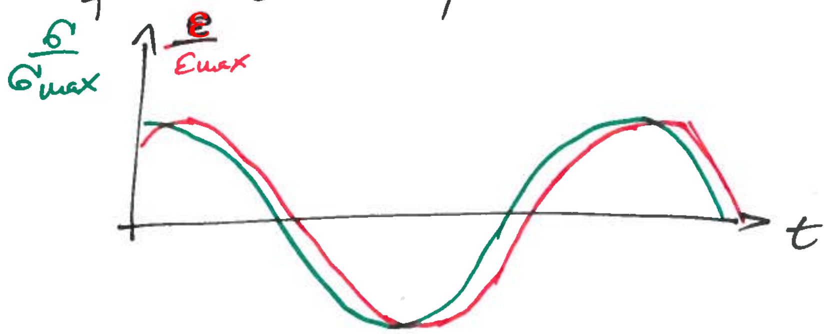
on long time scales elastic, on short time scales there is viscous creep

- Rheometer measure stress and strain relationships

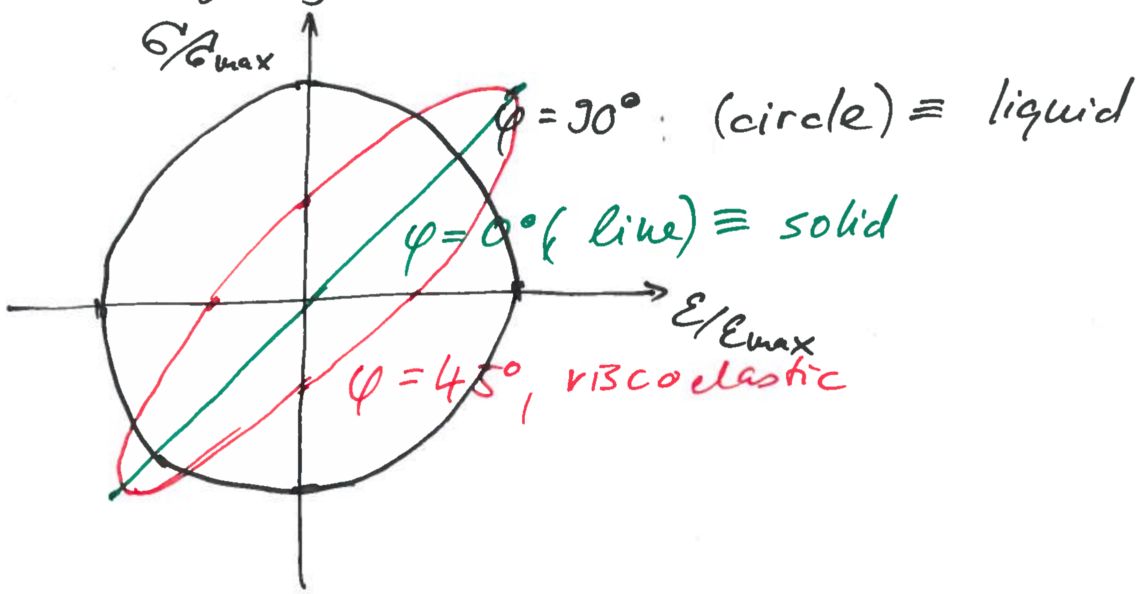


rotation speed $\propto$ shear rate
torque on unmoved part $\propto$ shear stress

- for oscillatory stresses/strains, we have

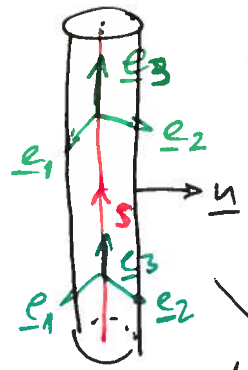


giving a stress-strain hysteresis

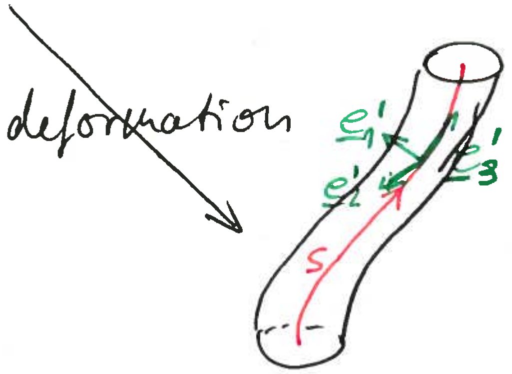


Rod theory

- Consider long and thin rods at small strains
- rods are parametrized by arc length s along its center line $\underline{r}(s)$



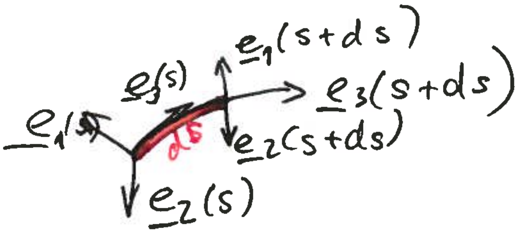
- a material frame keeps track of the deformation
- transverse stresses are considered zero $\underline{\sigma} \cdot \underline{n} = 0$ at the boundary



- $\underline{e}_3 = \partial_s \underline{r}$

change of material frame along the rod is given by vector $\underline{\Omega}(s)$ such that

$$d\underline{e}_i(s) = \underline{\Omega}(s) \times \underline{e}_i(s) \quad \left(\text{infinitesimal rotation by } \underline{\Omega}(s)\right)$$



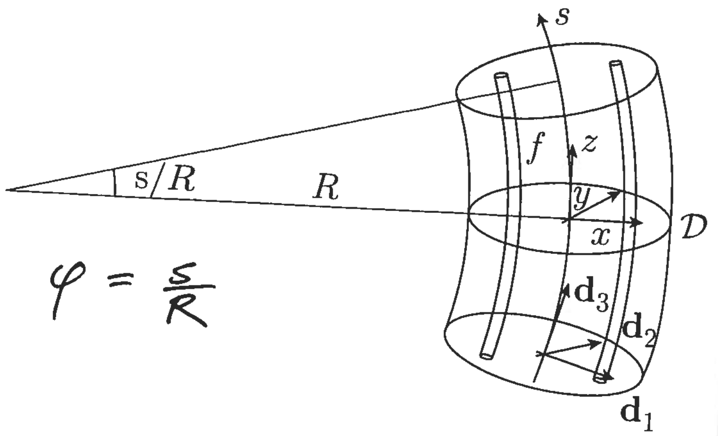
$$\underline{\Omega}(s) = \begin{pmatrix} \kappa_1 \\ \kappa_2 \\ \tau \end{pmatrix}$$

τ ... rotation of the cross-section around $\underline{e}_3$ $\equiv$ twist of the rod around central axis

κ_1, κ_2 ... rotation of rod tangent around $\underline{e}_1$ (for κ_1) or $\underline{e}_2$ (for κ_2) $\equiv$ bending of rod

Consideration of (small) uniform bending

- set cross-section under consideration into origin
- consider bending in $x-z$ plane



$$\varphi = \frac{s}{R}$$

At $z=0$, we have

$$\epsilon_{xx} = \frac{\partial u_x(x,y)}{\partial x}$$

$$\epsilon_{yy} = \frac{\partial u_y(x,y)}{\partial y}$$

$$\epsilon_{xz},\ \epsilon_{yz} = 0$$

$$dl_z = \varphi \cdot R$$

$$dl_z' = \varphi \cdot (R+x) \cdot (1+\eta)$$

$$\epsilon_{zz}(x) = \frac{dl_z' - dl_z}{dl_z}$$

- cross-section stay plane $\Longleftrightarrow$ there is mirror-symmetry with respect to reflection at the cross-section

$$= \left[\varphi \cdot (R+x)(1+\eta) - \varphi R\right] / \varphi \cdot R$$

$$= \eta \frac{\varphi \cdot R}{\varphi \cdot R} + \frac{\varphi \cdot x}{\varphi \cdot R} + \frac{x}{R} \cdot \eta$$

$$\approx \eta + \frac{x}{R} + \text{second order terms}$$

<u>stress</u>

$$\sigma_{zz} = E\,\epsilon_{zz} = E\left(\eta + \frac{x}{R}\right),$$

σ_{ix}, σ_{iy} vanish (negligible forces on boundary)

$$\epsilon_{xx} = \epsilon_{yy} = -\nu\left(\eta + \frac{x}{R}\right)$$

bending energy: $E_{bend} = \int \frac{\sigma_{ij}\,\varepsilon_{ij}}{2}\, dx\,dy\,ds$

$$= \int \frac{\sigma_{zz}\,\epsilon_{zz}}{2}\, dx\,dy\,ds$$

$$= \frac{E \cdot L}{2} \int \left(\eta + \frac{x}{R}\right)^2 dx\,dy$$

minimizing the bending energy with respect to y gives

$$0 \stackrel{!}{=} \frac{\partial E}{\partial y} = \frac{EL}{2}\int\left(\frac{x}{R} + y\right)dx\,dy$$

$$\Rightarrow \quad \frac{\langle x\rangle}{R} + y = 0$$

As $\langle x\rangle = 0$ be definition of the center line,

$$y = 0$$

→ the center line is not stretched in the rod and is a neutral line

For arbitrary (but small) bending

$$\frac{E_{bend}}{L} = \frac{E}{2}\iint\left(\kappa_2\, x - \kappa_1\, y\right)^2 dx\,dy$$

$\kappa_2 = -\frac{1}{R_1}$, $\kappa_1 = +\frac{1}{R_2}$, R_1, R_2 .. bending radii in x-z, or y-z plane, respectively

$$= \frac{E}{2}\int\Big(\kappa_2 x^2 + \kappa_1 y^2 \underbrace{- 2\kappa_2\kappa_1 xy}_{\text{integrates to zero, for appropriate choice of } e_1, e_2.}\Big)dx\,dy$$

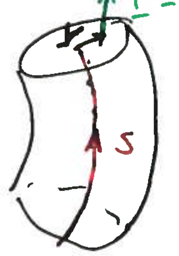
$$= \frac{E}{2}\Big[\kappa_1^2 \underbrace{\int x^2\,dx\,dy}_{=:I^{(1)}} + \kappa_2^2 \underbrace{\int y^2\,dx\,dy}_{I^{(2)}}\Big]$$

$I^{(1)}, I^{(2)}$ principal moments of inertia

for circular cross-section $I^{(1)} = I^{(2)} = \frac{\pi r^4}{4}$

- bending moment acting on a cross-section

$F = \sigma_{iz}(x,y)\, dx\, dy\, \underline{e}_i$



$$\underline{r} = x\, \underline{e}_1(s) + y\, \underline{e}_2(s)$$

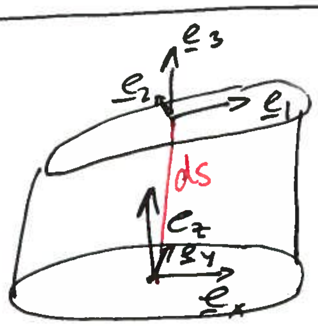
$$dM = \underline{r} \times \underline{F}$$

$$= (x\, \underline{e}_1 + y\, \underline{e}_2) \times \sigma_{zz}\, \underline{e}_3\, dx\, dy$$

$$\sigma_{zz} = -E\kappa_2 x + E\kappa_1 y$$

$$M = \int_{\text{cross section}} dM = E\kappa_1 I^{(1)} \underline{e}_1 + E\kappa_2 I^{(2)} \underline{e}_2$$

Consideration of uniform small twist of a rod



- center line remains undeformed
- cross sections get rotated along s
- we will see that cross-sections do not remain plane but get warped

- rotate coordinate system, such that

$\underline{e}_1(s) = \underline{e}_x$, $\underline{e}_2(s) = \underline{e}_y$

in cross section above at s+ds a point

$\underline{r}(x,y)$ got rotated to $\underline{r}'(x,y) = \underline{r}(x,y) + \underbrace{d\underline{\varphi} \times \underline{r}}$

where $d\underline{\varphi} = \tau \cdot ds \cdot \underline{e}_z$

$\Rightarrow u_x = -\tau z y$, $u_y = \tau z x$

- get the strain; by symmetry considrations one can infer that

$\epsilon_{xx}, \epsilon_{yy}, \epsilon_{yx}, \epsilon_{zz}$ are even functions of τ

$\Rightarrow O(\tau^2)$

only $\epsilon_{xz}, \epsilon_{yz} = O(\tau)$

(*) $\epsilon_{xz} = \frac{1}{2}(u_{z,x} + u_{x,z}) = \frac{1}{2}(u_{z,x} - \tau y)$

$\epsilon_{yz} = \frac{1}{2}(u_{z,x} + \tau \cdot x)$

$$\rightarrow \sigma_{x,z} = 2\mu\epsilon_{xz}$$

$$\sigma_{yz} = 2\mu\epsilon_{yz}$$

force equilibrium dictates

$$\sigma_{xz,x} + \sigma_{yz,y} = 0 \Rightarrow \epsilon_{xz,x} + \epsilon_{yz,y} = 0$$

$$\rightarrow \operatorname{rot}\begin{pmatrix} -\epsilon_{yz} \\ \epsilon_{xz} \\ 0 \end{pmatrix} = 0 \rightsquigarrow \begin{pmatrix} -\epsilon_{yz} \\ \epsilon_{xz} \\ 0 \end{pmatrix} = \underline{\nabla}\chi(x,y)$$

From (*)

$$\Delta\chi = -\epsilon_{yz,x} + \epsilon_{xz,y}$$

$$= -\frac{1}{2}\left(u_{z,yx} + \tau\right) + \frac{1}{2}\left(u_{z,xy} - \tau\right)$$

$$= -\tau$$

define $\bar{\chi} = \frac{\chi}{\tau}$, then $\Delta\bar{\chi} = -1$

boundary conditions $\sigma_{xz} n_x + \sigma_{yz} n_y = 0$

$\underline{n}$ ... normal vector $\Rightarrow \epsilon_{xz} n_x + \epsilon_{yz} n_y = 0$

$\underline{t}$ ... tangent on rod surface

$\rightsquigarrow$ this implies

$$\bar{\chi}_x t_x + \bar{\chi}_y t_y = 0 = \underline{\nabla}\bar{\chi}\cdot\underline{t}$$

$\rightarrow \bar{\chi}$ is constant along the boundary of the cross-section

we set $\bar{\chi}|_{\text{boundary}} = 0$

- circular cross-setion gives $\bar{\chi} = \frac{R^2 - (x^2+y^2)}{4}$

Energy of twist

$$\frac{E_{twist}}{L} = \frac{1}{2}\iint \sigma_{ij}\,\varepsilon_{ij}\,dx\,dy$$

$$= \frac{1}{2}\iint \left(2\sigma_{xz}\varepsilon_{xz} + 2\sigma_{yz}\varepsilon_{yz}\right)dx\,dy$$

$$= \frac{1}{2}\iint 4\mu\left(\varepsilon_{xz}^2 + \varepsilon_{yz}^2\right)dx\,dy$$

$$= 2\mu\tau^2 \underbrace{\iint \left(\overline{\chi}_x^{\,2} + \overline{\chi}_y^{\,2}\right) dx\,dy}_{J \text{ ... moment of twist}}$$

$$= 2\mu\tau^2 J$$

- just like $I^{(1)}$ & $I^{(2)}$, J is a geometrical invariant of the rod cross-section
- for circular cross-section $J = \frac{\pi}{2} a^4$

Moment of twist

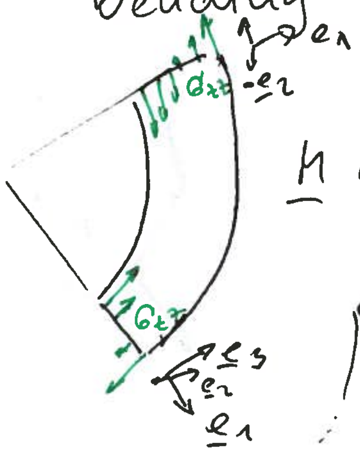
- twisting generates moments on a cross section

$\underline{M} \propto \underline{e}_3$, therefore

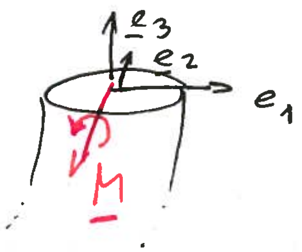
$$\underline{M} = \frac{dE_{twist}}{d\tau}\,\underline{e}_3 = \mu J \cdot \tau\,\underline{e}_3$$

short summary

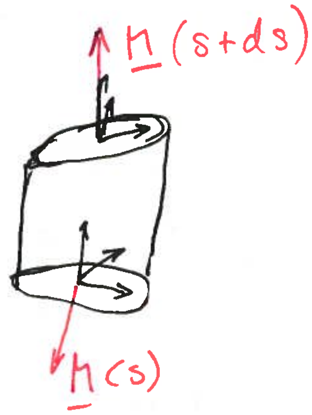
bending



$\underline{M} \propto -\underline{e}_2$



twist



no net force on cross section as

$$\int_{\text{cross section}} \sigma_{zz} = 0$$

Superposition of bending and twist

- strain, stress and moments of bending and twist can be superimposed, also elastic energy contributions
  $\Longrightarrow$ get a description for rods that are (weakly) bent and twisted
- up to now we had uniform twist and bending $\Longrightarrow$ results can be extended to slowly varying deformations
  (variation on a scale $\lambda \gg h$, h.. rod thickness)
- elastic energy of the rod is then taken as

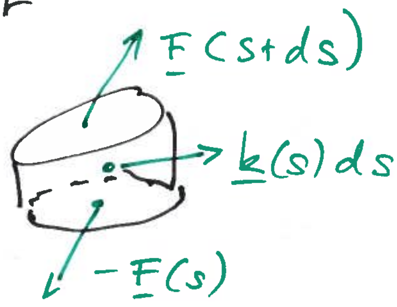
$$E_{rod} = \int ds \left(\frac{EI^{(1)}}{2} \kappa_1(s)^2 + \frac{EI^{(2)}}{2} \kappa_2(s)^2 + \frac{\mu J}{2} \tau(s)^2 \right)$$

Equilibrium condition for deformed rods

force acting on a cross section

$$\underline{F} = \int_{\substack{\text{cross}\\\text{section}}} \sigma_{i3}\, \underline{e}_i \, dx\, dy$$

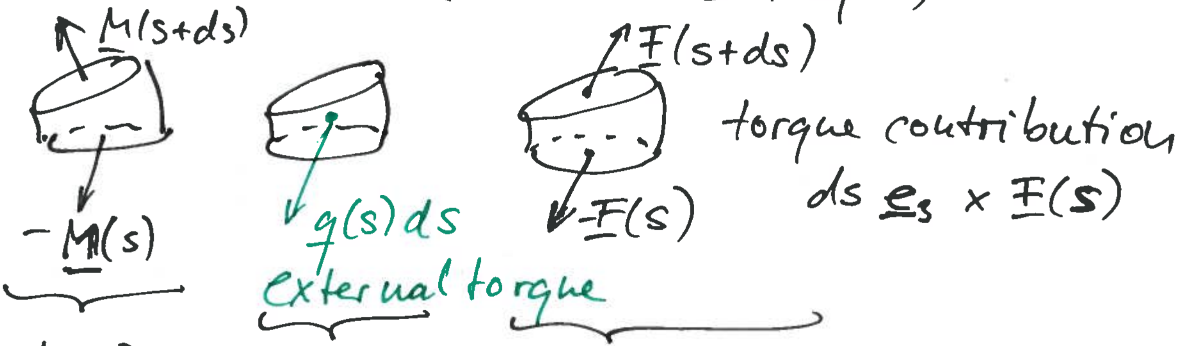
- total force on a length segment needs to vanish



$$(1) \quad \frac{\partial \underline{F}}{\partial s} + \underline{k}(s) = 0$$

where $\underline{k}(s)$ is external force per unit length

- total moment on a length segment needs to vanish (moment $\equiv$ torque)



$$(2) \quad \frac{\partial \underline{M}(s)}{\partial s} + q(s) + \underline{e}_3 \times \underline{F}(s) = 0$$

- mostly $q(s)$ is zero
- if $\underline{k}(s)$ is only non-vanishing in singular points, then (1) simplifies to

$$\underline{F} = \text{constant}$$

and (2) becomes

$$\underline{M}(s) + \underline{r} \times \underline{F}(s) = \text{const.}$$

$\underline{r}(s)$: center line of the rod

because $\underline{e}_3 = \partial_s \underline{r}$

- discuss the example of a circular rod, subject to bending in a plane

We had $\underline{M} = E\kappa_1 I^{(1)} \underline{e}_1 + E\kappa_2 I^{(2)} \underline{e}_2$

now $\partial_s \underline{e}_3 = \underline{\Omega} \times \underline{e}_3$, $\underline{\Omega} = (\kappa_1, \kappa_2\, \tau)$

$= \kappa_2 \underline{e}_1 + \kappa_1 \underline{e}_2$

and $\underline{e}_3 \times \partial_s \underline{e}_3 = \kappa_2 \underline{e}_2 + \kappa_1 \underline{e}_1$

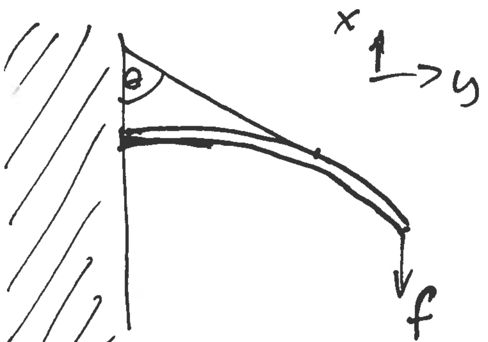
we also had $I^{(1)} = I^{(2)}$

$$\Rightarrow \underline{M} = E I^{(1)} (\kappa_1 \underline{e}_1 + \kappa_2 \underline{e}_2)$$
$$= E I^{(1)} (\underline{e}_3 \times \partial_s \underline{e}_3)$$
$$\overset{\underline{e}_3 = \partial_s \underline{r}}{=} E I^{(1)} (\partial_s \underline{r} \times \partial_s^2 \underline{r})$$

Equilibrium condition (2) gives

$$\partial_s \underline{M} + \underline{e}_3 \times \underline{F} = 0$$
$$E I^{(1)} (\partial_s \underline{r} \times \partial_s^3 \underline{r}) + \partial_s \underline{r} \times \underline{F} = 0 \qquad (*)$$

Consider cylindrical rod, clamped to a wall



with centerline

$\underline{r} = \begin{pmatrix} x(s) \\ y(s) \\ 0 \end{pmatrix}$, $\underline{F} = \begin{pmatrix} 0 \\ f \\ 0 \end{pmatrix}$ at the end point

Introduce angle $\theta(s)$ such that

$\partial_s \underline{r} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ .. tangent vector

$\partial_s^2 \underline{r} = \partial_s \theta \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$, $\partial_s^3 \underline{r} = \partial_s^2 \theta \begin{pmatrix} \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix}$
$- \partial_s \theta\, \partial_s \underline{r}$

$$\Rightarrow \partial_s \underline{r} \times \partial_s^3 \underline{r} = \partial_s^2 \theta \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix}$$
$$\partial_s \underline{r} \times \underline{F} = \begin{pmatrix} 0 \\ 0 \\ \sin\theta\, f \end{pmatrix}$$

(*) gives then

$$-IE\frac{\partial^2\theta}{\partial s^2} + f\sin\theta = 0$$

Integration gives

$$\frac{IE}{2}\left(\frac{\partial\theta}{\partial s}\right)^2 + f\cos\theta = c_1$$

at end of the rod, we have to have $\underline{M}(L)=0$

$$\rightarrow \partial_s\theta(L) = 0$$

$$\cancel{\frac{IE}{2}\cdot 0} + f\cos(\theta(L)) = c_1$$

further integration gives

$$s(\theta) = \sqrt{\frac{IE}{2f}}\int_{\theta(L)}^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{\cos\theta(L)-\cos\theta}}$$

as $s(\theta(L)) = L$,

$$L \stackrel{!}{=} \sqrt{\frac{IE}{2f}}\int_{\theta(L)}^{\pi/2}\frac{d\theta}{\sqrt{\cos(\theta(L))-\cos\theta}}$$

We have now $s(\theta)$, calculation gives $\theta(s)$,

As $\frac{dx}{ds} = \sin\theta$ $\quad x = \int_0^L \sin\theta\, ds$, $y = \int_0^L \cos\theta\, ds$

$\frac{dy}{ds} = \cos\theta$

calculation gives

$$x(\theta) = \sqrt{\frac{2IE}{f}}\left(\sqrt{\cos\theta_L} - \sqrt{\cos\theta_L - \cos\theta}\right.$$

$$y(\theta) = \sqrt{\frac{2IE}{f}}\int_0^{\frac{\pi}{2}}\frac{\cos\theta\, d\theta}{\sqrt{\cos\theta_L - \cos\theta}}$$