Stochastic resonance

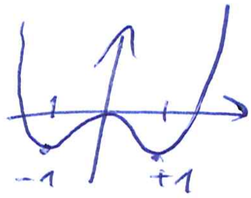
Motivational example:
Ice ages

- $x(t) \equiv$ global ice volume
  $|\tilde{x}(\omega)|^2$: peak at $\Omega = \frac{2\pi}{T}$, $T \approx 10^5 y$.
- external driving?
  eccentricity of earth orbit
  oscillates with $T = 0.96 \cdot 10^5 y$,
  induces $O(0.3K)$-temperature
  variation by change in
  irradiation
  $\longrightarrow$
  too weak to account for
  $\sim 10K$ temperature change
  during ice age.
- add noise!

Minimal model:

- diffusion in double-well potential with weak periodic driving.

$$U_0 = -\frac{1}{2}x^2 + \frac{1}{4}x^4.$$



$\Delta U = \frac{1}{4}$

$$U = U_0 + A\cos\Omega t.$$

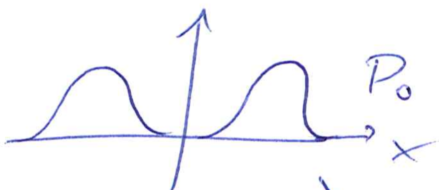
$$\dot{x} = -\frac{\partial U}{\partial x} + \xi, \qquad \langle \xi(t)\,\xi(t')\rangle = 2D\,\delta(t-t').$$

Kramer's rate

$$r_0 = \frac{1}{2\pi}\sqrt{|U''(0)|\,U''(1)}\,\exp\left(-\frac{\Delta U_0}{D}\right)$$

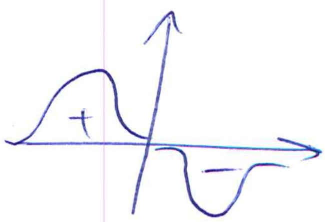
$$= \frac{\sqrt{2}}{2\pi}\exp\left(-\frac{1}{4D}\right)$$

Eigenfunctions of Fokker-Planck operator



$\lambda_0 = 0$

$\equiv$ steady-state distribution



$\equiv$ slowest mode

$\lambda_1 = -r_0 < 0.$

two-state (telegraph) process.

$$x=-1 \longleftrightarrow x=+1$$

$$r_-: \quad n_- \longrightarrow n_+$$

$$r_+: \quad n_- \longleftarrow n_+.$$

$$\dot{n}_+ = -\dot{n}_- = r_- n_- - r_+ n_+.$$

$$= r_- - \underbrace{(r_+ + r_-)}_{r.} n_+.$$

formal solution

$$n_+(t) = \left(\exp - \int_{t_0}^{t} dt' \, r\right) n_+(t_0)$$

$$+ \exp\left(-\int_{t_0}^{t} dt' r\right) \cdot \int_{t_0}^{t} dt' \, r_-(t') \cdot \exp \int_{t_0}^{t'} r \, dt''$$

Note

$$\frac{d}{dt}\left[\left(\exp \int^{t} r\right) n_+\right] = r_- \exp \int^{t} dt' \, r.$$

check:

- $n_+(t_0) = 1 \cdot n_+(t_0) + 1 \cdot 0.$
- $\dot{n}_+(t_0) = -r \quad n_+(t) +$
  $\exp\left(-\int_{t_0}^{t} dt' r\right) \cdot r_-(t) \cdot$
  $\exp\left(+\int_{t_0}^{t} dt'' r\right).$
  $= - \quad r \, n_+(t) + r_-$

For $A << D$,

$$r_\pm = \frac{\sqrt{2}}{2\pi} \cdot \exp\left(-\frac{1}{4D} \mp \frac{A}{D} \cos \Omega t\right)$$

$$\approx r_0 \cdot \left(1 \mp \frac{A}{D} \cos \Omega t\right).$$

$r_1 = \frac{r_0 A}{D}$

Insert this in equation for n_+

$$n_+ = \exp -2r_0(t-t_0) \cdot$$

$$\left[n(t_0) + \int_{t_0}^{t} dt' \left(r_0 + r_1 \cos \Omega t'\right) \exp 2r_0(t'-t_0)\right]$$

$$= \exp - 2r_0(t-t_0)$$

$$\left[n(t_0) + \frac{1}{2}\left(\exp 2r_0(t-t_0) - 1\right)\right.$$

$$+ \frac{r_1}{\sqrt{\Omega^2 + 4r_0^2}} \cos(\Omega t - \phi) \exp 2r_0(t-t_0)$$

$$\left. - \cos(\Omega t_0 - \phi)\right]$$

$$\tan \varphi = \frac{\Omega}{2r_0}.$$

Next: autocorrelation function

$\longrightarrow$

power spectrum.

$$\langle x(t)\rangle = x_+ n_+(t) + x_- n_-(t)$$
$$= n_+ - n_- \quad \text{for } x_\pm = \pm 1$$

$$\langle x(t)\cdot x(t+T)\,|x_0, t_0\rangle =$$

Markov property

$$= \sum_{s,s'} s s' P(s, t+T\,|s', t)\, P(s', t\,|x_0, t_0)$$

$$= n_+(t+T\,|+1, t)\cdot n_+(t\,|x_0, t_0)$$
$$- n_+(t+T\,|-1, t)\cdot n_-(t\,|x_0, t_0)$$
$$- n_-(t+T\,|+1, t)\cdot n_+(t\,|x_0, t_0)$$
$$+ n_-(t+T\,|-1, t)\cdot n_-(t\,|x_0, t_0).$$

- use $n_- = 1 - n_+$
- substitute above result
- limit $t_0 = -\infty$

$$\langle x(t)\,x(t+T)\,|x_0, t_0\rangle =$$
$$\exp(-2r_0T) + 2r_1 +$$
$$4r_1^2\left[-\exp(-2r_0T)\cos^2(\Omega t - \varphi) + \cos(\Omega(t+T) - \varphi)\cdot\cos(\Omega t - \varphi)\right].$$

⚠ explicite time-dependence, even for $t_0 \to -\infty$.

Reason: periodic driving defines a phase $\varphi = \Omega t$

- average over phase φ.

$$C(T) = \langle x(t)\, x(t+T) \,|\, x_0, t_0 \rangle_\varphi =$$

$$\underbrace{\exp(-2r_0|T|)\left[1 - 2r_1^2\right]}_{\text{decaying component}} + \underbrace{2r_1^2 \cos \Omega T}_{\text{periodic component}}$$

- Wiener-Khinchine theorem.

$S(\omega) = C(T)^\sim$ exists.

$\equiv$ power spectral density $S(\omega) \delta = \langle |\tilde{x}(\omega)|^2 \rangle$

<u>A physicist's proof</u>

$$\tilde{x}(\omega) = \int_{-\infty}^{\infty} dt\ x(t) \exp -i\omega t.$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\omega\ \tilde{x}(\omega) \exp i\omega t$$

$$C(T) = \langle x(t) \cdot x(t+T) \rangle_t = \left(\frac{1}{2\pi}\right)^2 \cdot \int_{-\infty}^{\infty} d\omega_1 \int_{-\infty}^{\infty} d\omega_2$$

$$\langle \tilde{x}(\omega_1) \tilde{x}(\omega_2) \rangle \langle \exp i(\omega_1 + \omega_2) t \rangle_t \cdot \exp i\omega_2 T.$$

$$= \left(\frac{1}{2\pi}\right)^2 \int_{-\infty}^{\infty} d\omega_1 \langle \tilde{x}(\omega_1) \tilde{x}(-\omega_1) \rangle \cdot \exp -i\omega_1 T$$

Example: spectral power density of Ornstein-Uhlenbeck process

$$T\dot{x} = -x + \xi .$$

$$(1 + Ti\omega)\tilde{x}(\omega) = \tilde{\xi}(\omega)$$

$$S_x(\omega) = \langle |\tilde{x}(\omega)|^2 \rangle$$

$$= \left|\frac{1}{1+Ti\omega}\right|^2 \langle |\tilde{\xi}(\omega)|^2 \rangle$$

$$= \frac{2D}{(T\omega)^2 + 1} .$$

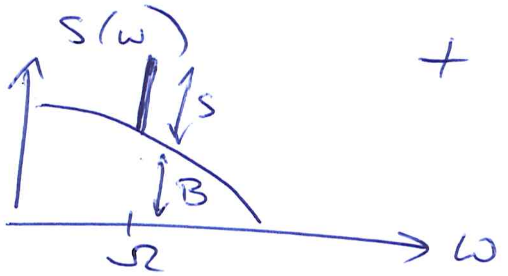
$\equiv$ Lorentzian

Power spectrum:

$$S(\omega) = (1 - 2r_1^2) \underbrace{\frac{4r_0}{4r_0^2 + \omega^2}}_{\text{Lorentzian shape}}$$

$$+ 2\pi r_1^2 \left(\delta(\omega - \Omega) + \delta(\omega + \Omega) \right).$$

$$\frac{\quad}{4r_0^2 + \omega^2}$$



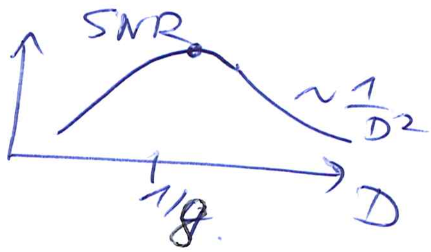
- $SNR = \frac{S}{B}$

Double-check: Which limit is taken here?

$$= \pi \frac{r_1^2}{r_0}$$

$$= \pi \cdot \left(\frac{r_0 A}{D}\right)^2 \frac{1}{r_0} \quad , \quad r_0 = \frac{\sqrt{2}}{2\pi} \exp\left(-\frac{1}{4D}\right)$$

$$= \frac{1}{\sqrt{2}} \frac{A^2}{D^2} \exp -\frac{1}{4D}.$$



- Spectral amplification factor

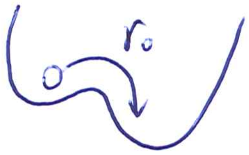
$$SPA = \frac{A_1(D)^2}{A^2}, \quad \langle x(t) \rangle = A_1(D) \cos(\Omega t - \varphi)$$

$$A_1(D) = \frac{A}{D} \frac{2r_0}{\sqrt{4r_0^2 + \Omega^2}}$$

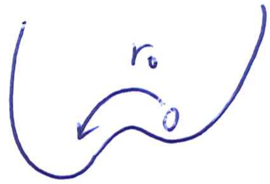
$$= \frac{4r_0^2}{D^2(4r_0^2 + \Omega^2)}$$

⇒ SPA attains maximum at $\Omega = 2 r_0$.

⇒ Intuitive interpretation



$$\frac{1}{r_0} \approx \frac{T}{2}$$



$$\frac{1}{r_0} \approx \frac{T}{2}.$$

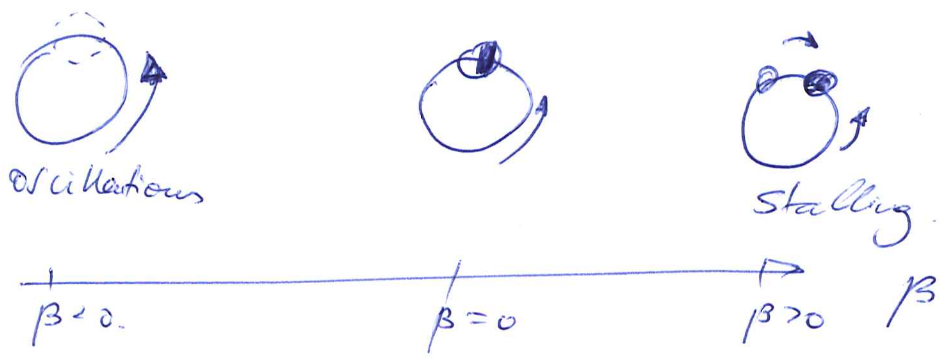
Biological exple

Mechano-receptors in crayfish

- predator = periodic pressure signal
  turbulence = noise.
- sensory detection optimal at physiological noise levels.
- experimental test by playing signal + ~~variable~~ noise. and recording neuron spiking

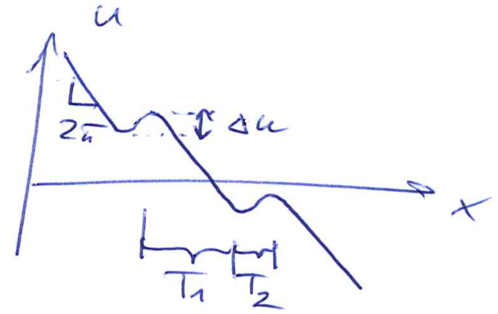
Coherence resonance

Recall: saddle-node bifurcation



Now with noise:

$$\dot{x} = -\frac{\partial u}{\partial x} + \xi$$



period: $T = T_1 + T_2 \equiv$ random variable

- T_1: $\dot{x} = 2\pi + \xi$, $\langle \xi(t)\xi(t')\rangle = 2D\,\delta(t-t')$
  $\langle T_1\rangle \approx 1$, $\langle T_1^2\rangle - \langle T_1\rangle^2 \approx 2D$.
- T_2: Poisson point process: exponential distribution of waiting times
  $$\langle T_2\rangle = \frac{1}{r}, \quad \langle T_2^2\rangle - \langle T_2\rangle^2 = \frac{1}{r}$$
  $$r = \frac{1}{a}\exp\left(-\frac{b}{D}\right).$$
- $\langle T^2\rangle - \langle T\rangle^2 = 2D + a\exp\left(+\frac{b}{D}\right)$