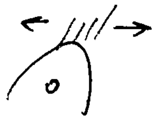
Nonlinear dynamics of active oscillators

Examples of active, biological oscillators.

(1) circadian rhythm $T = 24h$

(2) single heart muscle cell $T = 1s$.

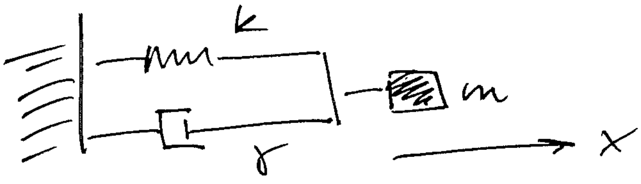
(3) hair bundle in the inner ear $T = 0.1ms$



$\Rightarrow$ active oscillators that oscillate spontaneously.

Today: Minimal model for active oscillators.

Example of passive oscillator: harmonic oscillator



m … mass
k … spring constant
γ … friction coefficient

force balance:

$$\underset{\text{inertia}}{m\ddot{x}} + \underset{\text{friction}}{\gamma \dot{x}} + \underset{\text{elasticity}}{k x} = 0$$

- Solutions are damped oscillations

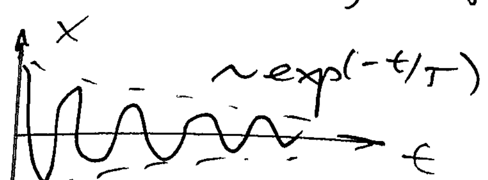
(*) $$x(t) = x(0) \exp(-t/T) \cos \omega_0 t$$

with

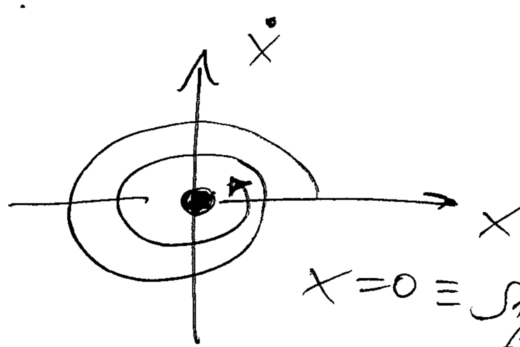
$$T = m/2\gamma \quad \text{… damping time}$$

$$\omega_0 \approx \sqrt{k/m} \quad \text{… resonance freq.}$$

for $\gamma^2 \ll m \cdot k$



Phase dynamics.



$x = 0 \equiv$ stable fixed point.

Exercise: check (*).

Thought experiment:

Q: What happens for $\gamma < 0$?

A: Oscillation amplitude grows as $\exp(+t/\tau)$.

This requires energy input.

Active oscillators can be described by nonlinear friction $\gamma = \gamma(x)$ that can become negative.

Example: Van-der-Pol oscillator.

Choice $\gamma = \mu(\lambda - 2x^2)$ yields

(**) $$m\ddot{x} + \mu(\lambda - 2x^2)\dot{x} + kx = 0.$$

Ansatz:

$$x(t) \approx a \cos\omega_0 t.$$

What sign has γ?

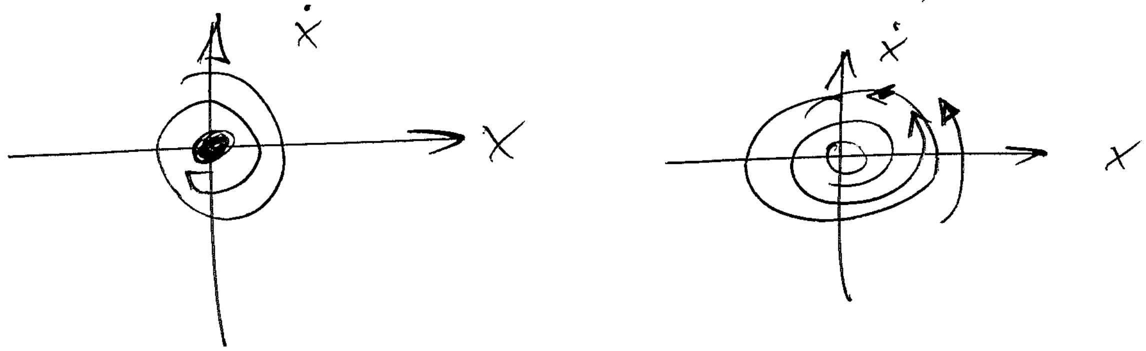
Trick: average over one oscillation cycle $\Rightarrow$ $\langle\gamma\rangle = \mu(\lambda - a^2)$.

$\Rightarrow$
- $\langle\gamma\rangle < 0$ for $a < \lambda^{1/2}$ $\Rightarrow$ amplitude grows
- $\langle\gamma\rangle > 0$ for $a > \lambda^{1/2}$ $\Rightarrow$ amplitude decays

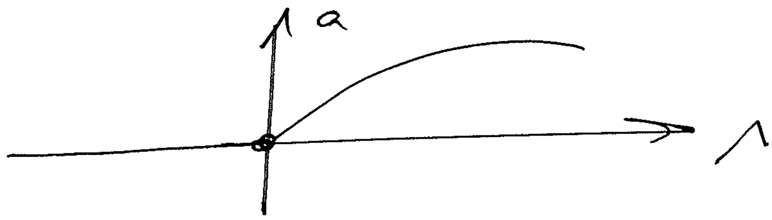
$\Rightarrow$ stable limit cycle oscillations with $a = \lambda^{1/2}$, $\omega_0 \approx \sqrt{k/m}$ if $\lambda > 0$.

Phase dynamics

$\Lambda < 0$ (passive) $\qquad\qquad$ $\Lambda > 0$ (active)



The onset of spontaneous oscillations for $\Lambda \geq 0$. corresponds to a Hopf bifurcation with control parameter Λ.



This transition from a stable fixed point $a = 0$ for $\Lambda < 0$ to spontaneous oscillations for $\Lambda > 0$ is called a Hopf bifurcation.

Normal form of Hopf bifurcation

(*). $\dot{Z} = i(\omega_c - \omega_1 |Z|^2)Z + \mu(\Lambda - |Z|^2)Z$.

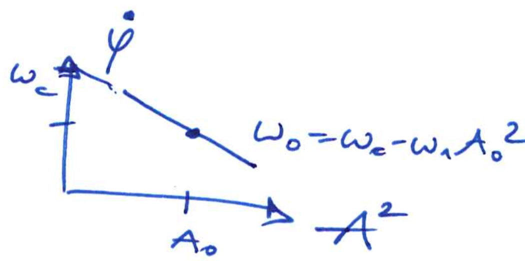
with complex oscillator variable

$Z = A \exp i\varphi$.

We can rewrite (*) as

$$\dot{A} = \mu(\Lambda - A^2)A.$$
$$\approx \mu A_0{}^2(A_0 - A) \quad \text{if } |A - A_0| \ll A_0 = \Lambda$$

$\dot{\varphi} = \omega_c - \omega_1 A^2$



$\omega_1 \equiv$ non-isochrony.

<u>Comment</u>:

How to construct Z?

For periodic signal $x(t) = A\cos\omega_0 t$.

Set $Z = x(t) + i\dot{x}(t)/\omega_0$.

Q: What happens at $\Lambda = 0$?

A: The oscillator is highly sensitive to external driving at the resonance frequency

$$m\ddot{x} + \mu(0 - 2\mu x^2)\dot{x} + kx = f\cos\omega_0 t.$$

ext. driving.

Ansatz: $x(t) = a\sin\omega_0 t$.

$$-2\mu a^2 \underbrace{\sin^2\omega_0 t}\cdot\omega_0 a\cdot\cos\omega_0 t = f\cos\omega_0 t$$

$\Rightarrow$ averages to $1/2$

$$\mu\omega_0 a^3 = f.$$

$$a \sim f^{+1/3}$$

$$\text{gain} = \frac{a}{f} \sim f^{-2/3}.$$

Application:

The dynamics of hair bundles in our inner ear can be mapped onto (**).

Hair bundles are tuned at $\Lambda = 0$ to amplify faint sounds (low f) more than loud sounds (high f).

Poincaré - Bendixon - Theorem.

Existence of closed orbits in 2D.

$(\dot{x}, \dot{y}) = f(x,y)$, $f \in C^1$

If $R \subseteq R^2$ is a trapping region

- closed, bounded
- no trajectory can leave R

then R contains either a fixed point or a closed orbit.

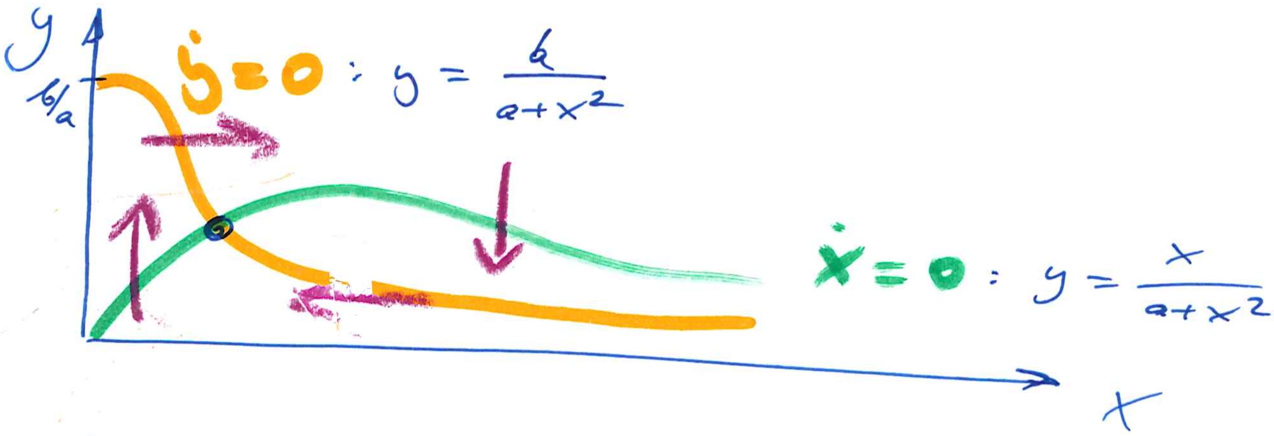
Example:

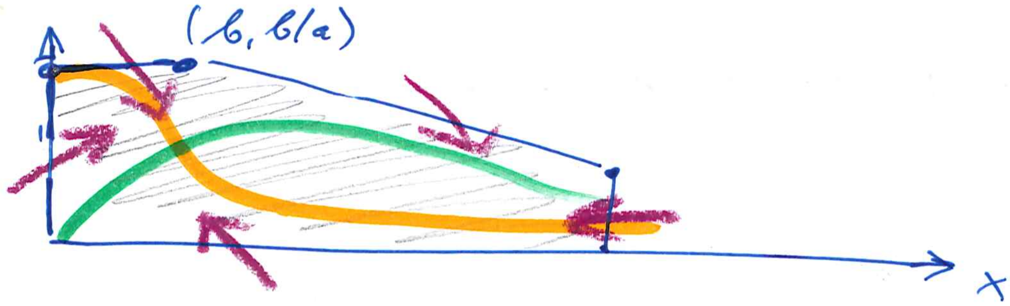
Glycolysis model (Sel'kov, 1968)

$x \equiv [\text{ADP}]$

$y \equiv [\text{Fructose-6P}]$.

$$\dot{x} = -x + ay + x^2 y$$
$$\dot{y} = b - ay - x^2 y$$





<u>fixed point</u>

$$x^* = b, \quad y^* = \frac{b}{a+b^2}.$$

$$\underline{\underline{J}} = \begin{pmatrix} -1+2x^*y^* & a+x^{*2} \\ -2x^*y^* & -(a+x^{*2}) \end{pmatrix}$$

$\Rightarrow$

$$\Delta = a+b^2 > 0.$$

$$T = \frac{-b^4+(2a-1)b^2+(a+a^2)}{a+b^2}.$$

$T \gtrless 0 \Leftrightarrow$ unstable / stable.