Biological Hydrodynamics

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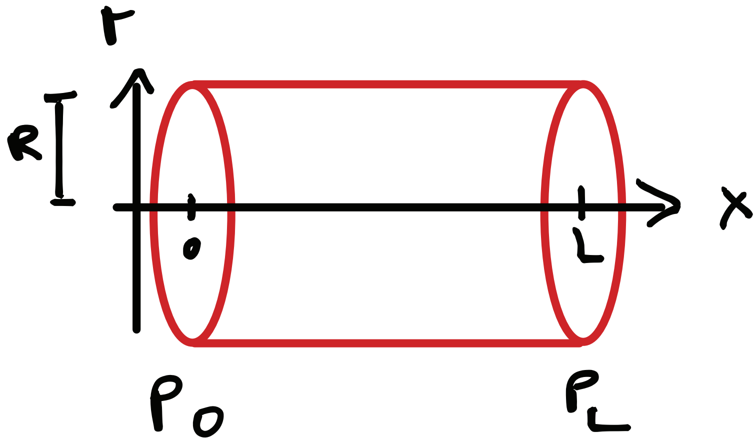
Next Tutorial: Thursday **28th November**, 14:50 - 16:20, MPI PKS Seminar Room 3

Tutorial 5: Poiseuille flow



Consider blood flow in a cylindrical vein. For simplicity, we treat blood as an incompressible Newtonian fluid with viscosity η. The vein has length L and radius R, and the pressures P_0 and P_L are applied respectively to the entry and the exit of the vein. The flow is axisymmetric, and subject to no-slip boundary conditions such that the fluid velocity is zero at the wall.

1. The velocity gradient tensor in cylindrical coordinates is given by

$$u = \begin{pmatrix} \partial_r v_r & \frac{1}{2}\left(\partial_r v_\phi - \frac{v_\phi}{r} + \frac{1}{r}\partial_\phi v_r\right) & \frac{1}{2}\left(\partial_x v_r + \partial_r v_x\right) \\ \frac{1}{2}\left(\partial_r v_\phi - \frac{v_\phi}{r} + \frac{1}{r}\partial_\phi v_r\right) & \frac{1}{r}\left(\partial_\phi v_\phi + v_r\right) & \frac{1}{2}\left(\frac{1}{r}\partial_\phi v_x + \partial_x v_\phi\right) \\ \frac{1}{2}\left(\partial_x v_r + \partial_r v_x\right) & \frac{1}{2}\left(\frac{1}{r}\partial_\phi v_x + \partial_x v_\phi\right) & \partial_x v_x \end{pmatrix} \tag{1}$$

   Taking into account the above assumptions, give the expression of the stress tensor in the vein.

2. Write the force balance equations, assuming the liquid flows at low Reynolds number.

3. Using force balance, show that the pressure gradient is uniform in the vein and give its expression in terms of the pressure at the boundaries.

4. Solve for the velocity profile $v_x(r)$.

5. You just ran 500m to the Mensa to get food. To alleviate the extra need of oxygen, would it be better to increase blood vessel radius by 10% or to increase the pressure difference by 30%? To answer this question, calculate the volumetric flow rate Q (the total volume of blood per unit time going through a section of the blood vessel). Show that it can be written as $Q = K(P_0 - P_L)$, where K is the hydraulic permeability of the vein.

6. Using an analogy with electrostatics, what is the hydraulic permeability of N tubes in parallel?

A swimmer is moving along at a velocity U in a fluid of viscosity η and density ρ.

7. Assuming he stops swimming, what distance will he cover before he stalls? Calculate this distance for a human swimmer (density $\rho_s \simeq \rho$, length $L = 2\,\mathrm{m}$ and velocity $U = 1\,\mathrm{m\,s^{-1}}$) and the bacterium *E. coli* (density $\rho_s \simeq \rho$, length $L = 2\,\mu\mathrm{m}$ and velocity $U = 2\,\mu\mathrm{m\,s^{-1}}$), both moving in water ($\eta = 10^{-3}\,\mathrm{Pa\,s}$).
   Hint: The viscous and inertial forces acting on the swimmer scale as $F_{\text{viscous}} \sim \eta U L$ and $F_{\text{inertial}} \sim \rho U^2 L^2$ respectively.